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A Study of Anti-rectangular AG-groupoids

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Abstract. An AG-groupoid H satisfying the rule of left semisymmetry, $(ab)a = b$ for all $a, b \in H$ is called an anti-rectangular AG-groupoid. This article is devoted to the study of various characterizations of anti-rectangular AG-groupoids and to the relations of this subclass with various subclasses of AG-groupoids and with other algebraic structures such as semigroups, commutative semigroups, monoids, abelian groups, etc. Furthermore, we remove the miss conception about the proper ideals of anti-rectangular AG-groupoids in [20] by proving that it is simple.

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1. Introduction

In 1972, Kazim and Naseeruddin introduced a new algebraic structure, based on left invertive law: $(ab)c = (cb)a$, by introducing braces to the left of the ternary commutative law: $abc = cba$. They called this structure a left almost semigroup or in short an LA-semigroup [13]. Later on, this structure is called upon by various names such as Abel-Grassmann's groupoid (abbreviated as AG-groupoid), left invertive groupoid [6] and right modular groupoid [3]. A groupoid satisfying $a(bc) = c(ba)$ (known as right invertive law) is known as right almost semigroup (RA-semigroup) or right Abel-Grassmann groupoid

(RA-groupoid). A groupoid which simultaneously satisfying the left and right invertive laws is known as self-dual AG-groupoid [31]. AG-groupoid is a well worked area of research having a variety of applications in various fields like matrix theory [1], flocks theory [13], geometry [31] and topology [20] etc.

Various aspects of AG-groupoids are investigated by different researchers and many results are available in literature (see, e.g., [8, 11, 12, 23, 32, 24, 16, 15, 28] and the references herein). Some new classes of AG-groupoids are discovered and investigated in [10, 30, 17, 9, 2, 37]. Protić and Stevanović [29] introduced anti-rectangular AG-band and proved that an anti-rectangular AG-band is anti-commutative. Mushtaq et al. introduced locally associative LA-semigroup [21]. The same idea was broaden by Mushtaq and Iqbal [19]. Iqbal et al. [9] introduced cyclic associative AG-groupoid and studied its various characterization [11, 12]. Mushtaq [22] introduced the concept of zeroids and idempoids in AG-groupoid. It has been proved that every AG-3-band is locally associative, but the converse is not true [23]. In the same paper, the authors decomposed AG-3-band and proved that AG-3-band has associative powers, i.e. for all $b,\, c$ in an AG-3-band, $bb^n = b^{n+1}$, $b^m b^n = b^{m+n}$, $(bc)^n = b^n c^n$ and $(b^m)^n = b^{mn}$ for all $m,\, n \in \mathbb{Z}^+$. They proved that an AG-3-band is fully semiprime and fully idempotent. They also proved that left and right ideal in AG-3-band coincide, and for two ideals B and C of an AG-3-band BC and CB are connected sets.

Shah et al. [32] introduced the notion of quasi-cancellativity in AG-groupoids and proved that every AG-band is quasi-cancellative. The same authors [33] introduced some new subclasses of AG-groupoids, namely: anti-commutative AG-groupoids, transitively commutative AG-groupoids, self-dual AG-groupoids, unipotent AG-groupoids, left alternative AG-groupoids, right alternative AG-groupoids, alternative AG-groupoids and flexible AG-groupoids, while M. Shah [31] introduced a new class of groupoids called Bol* groupoids and a new class of semigroups, called AG-groupoid semigroups.

Rashad [**?**] discussed some decompositions of locally associative left abelian distributive and Stein AG-groupoids and proved that (i) every locally associative left abelian distributive AG-groupoid K has associative powers, i.e. for all $b \in K$ and $m,\, n \in \mathbb{Z}^+$, (i) $bb^{n+1} = b^{n+1}b$ (ii) $b^m b^n = b^{m+n}$. Moreover, if an AG-groupoid K is Stein, then $b^n c^m = c^m b^n$ for every $c \in K$, where $m,\, n > 1$. Kamran [14] introduced the notion of AG-groups, defined cosets, factor AG-groups and proved Lagranges Theorem for AG-groups.

The concept of ideal in AG-groupoid was innovated by Mushtaq et al. in 2006 [24]. They also explored (left/right) ideals, connectedness and minimal (left/right) ideals in AG*-groupoid and in AG-band [25]. Iqbal and Ahmad [8] extended the concept of (left/right) ideals to cyclic associative AG-groupoids. Khan and Asif [16] studied various types of fuzzy ideals and characterized these ideals by their properties. Kehayopulu et al. [15] considered fuzzy ordered AG-groupoids as the generalization of fuzzy ordered semigroups.

2. Preliminaries

A groupoid H is called an AG-groupoid if it satisfies the left invertive law, $(ab)c = (cb)a$ for all $a, b, c \in H$ [27] and is called semigroup if it satisfies the associative law, $a(bc) = (ab)c$. The medial law, $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in H$, always holds in AG-groupoid. It is easy to prove that an AG-groupoid having left identity also satisfies the

paramedial law, $(ab)(cd) = (ac)(bd)$. If H is an AG-groupoid and $a, b, c, d \in H$, then H is called...
... self-dual if $a(bc) = c(ba)$ (called right invertive law),
... AG-2-band (in short AG-band) [22] if $c^2 = c$ for every c in H,
... AG-3-band if $(bb)b = b(bb) = b$ [35]. Every AG-band is AG-3-band [31],
... locally associative [34] if $(aa)a = a(aa)$,
... flexible [33] if $(ab)a = a(ba)$,
... transitively commutative if $ab = ba$ and $bc = cb \Rightarrow ac = ca$ [31],
... T_l^3 (resp., T_r^3) if $ba = bc \Rightarrow a = c$ (resp., $ab = cb \Rightarrow a = c$),
... T^3 if it is both T_l^3 and T_r^3 [10],
... T_f^4 (resp., T_b^4) if $ab = cd \Rightarrow ad = cb$ (resp., $ab = cd \Rightarrow da = bc$),
... T^4 if it is both T_f^4 and T_b^4 [10],
... quasi-cancellative [32] if $b^2 = bc$, $c^2 = cb$ and $b^2 = cb$, $c^2 = bc$ both implies $b = c$,
... anti-commutative if $bc = cb \Rightarrow b = c$ [31],
... left regular (resp., right regular) if $ca = cb \Rightarrow da = db$ (resp., $ac = bc \Rightarrow ad = bd$) and is regular if it is both left and right regular [**?**],
... left abelian distributive (LAD), (right abelian distributive (RAD)) if $a(bc) = (ab)(ca)$, (resp., $(bc)a = (ab)(ca)$),
... abelian distributive (AD) if it is both RAD and LAD [**?**],
... right (resp., middle/left) nuclear square if $(ab)\,c^2 = a\left(bc^2\right)$ $\left(\text{resp., } \left(ab^2\right)c = a\left(b^2c\right)/a^2\,(bc) = \left(a^2b\right)c\right)\right)$ [7],
... left commutative (LC) (resp., right commutative (RC)) if it satisfies $ab \cdot c = ba \cdot c$ $(a \cdot bc = a \cdot cb)$,
... bi-commutative (BC) if it is both LC and RC [30],
... left distributive (resp., right distributive) if $a \cdot bc = ab \cdot ac$ $(ab.c = ac.bc)$ [17],
... unipotent if $a^2 = b^2$ [17].
An element b is called right (resp., left) cancellative, if $ab = cb\,(ba = bc) \Rightarrow a = c$. It is called cancellative, if it is simultaneously right and left cancellative. An AG-groupoid is (right/left) cancellative if every element is (right/left) cancellative [7]. A non empty AG-groupoid H is quasigroup if the equations $hx = k$ and $xh = k$ have exactly one solution for all $h,\ k,\ x \in H$. A quasigroup with neutral element is called a loop, i.e. if there exists an element $e \in H$ such that $eb = be = b$ for every b in H [4]. A loop H is called right Cheban (resp., left Cheban) if $(c \cdot ba) = ca \cdot ab$ $(a(ab \cdot c) = ba \cdot ac)$ and is called Cheban loop if it is right and left Cheban loop [26]. A subset B of an AG-groupoid H is called right ideal (resp., left ideal) of H if $BH \subseteq B$ $(HB \subseteq B)$ and is called an ideal if it is both right and left ideal [25]. An AG-groupoid is called simple, if it has no proper ideal [25].

3. Anti-rectangular AG-groupoids

In this section, a new subclass of AG-groupoids is introduced and its existence is shown by producing various examples. It is interesting to note that out of the total 3 AG-groupoids of order 2, only 1 is anti-rectangular and which is associative. Out of all the total 20 AG-groupoids of order 3, none is anti-rectangular and out of the total 331 AG-groupoids of order 4 there are only 2 anti-rectangular AG-groupoids, in which one is associative and the other is non-associative and non-commutative. A complete table up to order 6 is presented at the end of this section.

Definition 3.1. *An AG-groupoid H that satisfies the identity $ab \cdot a = b$ (known as the rule of left semisymmetry* [4, page 58]*) for all $a, b \in H$ is called anti-rectangular* [18].

The following example depicts the existence of anti-rectangular AG-groupoid.

Example 3.2. *Cayley's Table 1 represents an anti-rectangular AG-groupoid of order* 8.

$\cdot$	a	b	c	d	x	y	z	t
a	c	d	a	y	b	t	x	z
b	x	b	y	a	d	z	c	t
c	a	z	c	x	t	b	y	d
d	b	x	t	z	c	a	d	y
x	z	a	d	b	y	x	t	c
y	d	c	z	t	x	y	b	a
z	t	y	b	d	a	c	z	x
t	y	t	x	c	z	d	a	b

Table 1

Clearly, if H is anti-rectangular AG-groupoid, then $H^2 = H$.

Enumeration, classification and construction of various algebraic structures is a well worked area of abstract algebra. Associative structures such as semigroup and monoid have been enumerated up to order 9 and 10 respectively. Non-associative structures: loop and quasigroup have been enumerated up to order 11. Shah et al. [31] used GAP [5] for enumeration of AG-groupoids up to order 6. Using GAP we also enumerate anti-rectangular AG-groupoids up to order 6 and further categorize them into commutative, non-commutative, associative and non-associative anti-rectangular AG-groupoids.

AG-groupoid/order	2	3	4	5	6
AG-groupoids	3	20	331	31913	40104513
Anti-rectangular AG-groupoids	1	0	2	0	0
Commutative anti-rectangular AG-groupoids	1	0	1	0	0
Non-commutative anti-rectangular AG-groupoids	0	0	1	0	0
Associative anti-rectangular AG-groupoids	1	0	1	0	0
Non-associative anti-rectangular AG-groupoids	0	0	1	0	0

Table 2: Enumeration of anti-rectangular AG-groupoid up to order 6

4. Anti-rectangular Test

Protić and Stevanović introduced a procedure for testing a finite Cayley's table (name after the British Mathematician Arthur Cayley $(1821-1895)$) for left invertive law [28]. Rashad [**?**] introduced different tests for verification of various subclasses of AG-groupoids. Iqbal [9] and Aziz [2] introduced tests respectively for verification of cyclic associative and self-dual AG-groupoids respectively. In the following we also provide a procedure to test a finite Cayley's table for anti-rectangular property.

Define two binary operations $\star$ and $\diamond$ on AG-groupoid $(H, \cdot)$ by:

$$\begin{aligned} x \star y &= ax \cdot a, \\ x \diamond y &= x, \end{aligned}$$

for any fixed element $a \in H$ and every $x, y \in H$.
To test an arbitrary AG-groupoid $(S, \cdot)$ for anti-rectangular property, it is sufficient to check that $x \star y = x \diamond y$ for all $x, y \in H$ and every $a \in H$. The tables for operation $\star$ can be constructed by writing a-row of the "$\cdot$" table as an index column for $\star$ table and operate its elements by a from the right. The tables for $\diamond$ can be obtained by writing the index row and than operate it by the index column. If for all $a \in H$ the tables of the operations $\star$ and $\diamond$ coincide, then $(ab)a = b$. Consequently, H is anti-rectangular. To illustrate the procedure take the following examples.

Example 4.1. *Let* $H = \{1, 2, 3, 4\}$. *Then* $(H, \cdot)$ *with the following Cayley's table is an AG-groupoid.*

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Table 3

To check anti-rectangular property for the given Table-3, we extend it in the way as described above.

.	1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4
1	1	3	4	2		1	1	1	1		1	1	1	1		1	1	1	1		1	1	1	1
2	4	2	1	3		2	2	2	2		2	2	2	2		2	2	2	2		2	2	2	2
3	2	4	3	1		3	3	3	3		3	3	3	3		3	3	3	3		3	3	3	3
4	3	1	2	4		4	4	4	4		4	4	4	4		4	4	4	4		4	4	4	4
							1					2					3					4		
					1	1	1	1	1	4	1	1	1	1	2	1	1	1	1	3	1	1	1	1
					3	2	2	2	2	2	2	2	2	2	4	2	2	2	2	1	2	2	2	2
					4	3	3	3	3	1	3	3	3	3	3	3	3	3	3	2	3	3	3	3
					2	4	4	4	4	3	4	4	4	4	1	4	4	4	4	4	4	4	4	4

Extended Table 3

From the extended Table 3, it is clear that the upper tables for the operation $\diamond$ and lower tables for $\star$ on the right-hand side of the original table coincide for all a in H, thus the AG-groupoid in Table 3 is an anti-rectangular AG-groupoid.
The following example depicts that this test fails for AG-groupoids which are not anti-rectangular.

Example 4.2. *Consider the AG-groupoid shown in the following Table* 4.

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	4	3	1	2
4	3	4	2	1

Table 4

Extending Table 4 in the way as described above we get the following.

.	1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4
1	1	2	3	4		1	1	1	1		1	1	1	1		1	1	1	1		1	1	1	1
2	2	1	4	3		2	2	2	2		2	2	2	2		2	2	2	2		2	2	2	2
3	4	3	1	2		3	3	3	3		3	3	3	3		3	3	3	3		3	3	3	3
4	3	4	2	1		4	4	4	4		4	4	4	4		4	4	4	4		4	4	4	4
							1					2					3					4		
					1	1	1	1	1	2	1	1	1	1	4	2	2	2	2	3	2	2	2	2
					2	2	2	2	2	1	2	2	2	2	3	1	1	1	1	4	1	1	1	1
					3	4	4	4	4	4	4	4	4	4	1	3	3	3	3	2	3	3	3	3
					4	3	3	3	3	3	3	3	3	3	2	4	4	4	4	1	4	4	4	4

Extended Table 4

We see that H is not anti-rectangular as the respective upper tables for $\diamond$ and lower tables for $\star$ on the right-hand side of the original table do not coincide.

5. Relation of Anti-rectangular AG-groupoid with other Subclasses of AG-groupoid

In the following we establish various relations of anti-rectangular AG-groupoid with other known subclasses of AG-groupoid like AG-3-band, cancellative, transitively commutative, regular and T_f^4-AG-groupoids, and with other algebraic structures, like semigroup, commutative semigroup, group and abelian group etc. We further check the converse of these relations and provide counterexample if it is not true.

Lemma 5.1. [2] *Let H be an anti-rectangular AG-groupoid. Then each of the following is true.*
(i) *H is self-dual,*
(ii) *H is AG-3-band.*

Proof. Let H be an anti-rectangular AG-groupoid and $a, b, c \in H$. Then

(i) By left semisymmetry and left invertive law we have,

$$a(bc) = (ba \cdot b)(bc) = (bc \cdot b)(ba) = c(ba) \Rightarrow a(bc) = c(ba).$$

(ii) By left semisymmetry, medial law, Part (i) and left invertive law we have,

$$\begin{aligned} (aa)a &= (aa)(aa \cdot a) = (a \cdot aa)(aa) = a(a(a \cdot aa)) \\ &= a(aa \cdot aa) = a((aa \cdot a)a) = a(aa). \end{aligned}$$

Now, by left semisymmetry $(aa)a = a$. Thus $a(aa) = (aa)a = a$. Hence H is AG-3-band. $\square$

However, the converse of the above lemma is not valid, as verified in the following.

Example 5.2. *Table* 5 *represents a self-dual AG-groupoid of order* 5*, which is not an anti-rectangular as* $(4 \cdot 5)4 \neq 5$*. AG-3-band of order* 4 *is given in Table* 6*, as* $(1 \cdot 2)1 \neq 2$*, thus it is not anti-rectangular.*

$\cdot$	1	2	3	4	5
1	1	3	4	2	1
2	4	2	1	3	4
3	2	4	3	1	2
4	3	1	2	4	3
5	1	3	4	2	1

Table 5

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	2	3	1
3	1	3	2	1
4	1	1	1	4

Table 6

As every AG-3-band is (i) locally associative [23], (ii) T^3-AG-groupoid [37] (iii) flexible [33] and (iv) has associative powers [23]. Thus, we have the following.

Corollary 5.3. *Every anti-rectangular AG-groupoid is*
(i) *locally associative,*
(ii) T^3*-AG-groupoid,*
(iii) *flexible,*
(iv) *has associative powers.*

Theorem 5.4. *For an anti-rectangular AG-groupoid* H*, any of the following hold:*
(i) H *is cancellative,*
(ii) H *is transitively commutative,*
(iii) H *is regular,*
(iv) H *is* T^4_f.

Proof. Let H be an anti-rectangular AG-groupoid and a , b, c, $d \in H$. Then

(i) To prove that H is left cancellative, let $ab = ac$. Then $(ab)a = (ac)a$, this by left semisymmetry implies $b = c$. Thus H is left cancellative. Again, let $ba = ca$. Then $a(ba) = a(ca)$. As by Corollary 5.3 H is flexible, thus $(ab)a = (ac)a$, this by left semisymmetry implies $b = c$. Thus H is right cancellative and hence cancellative.

(ii) Let $ab = ba$ and $bc = cb$. Then using left semisymmetry, Corollary 5.3 and Lemma 5.1

$$\begin{aligned} ac &= (ba \cdot b)c = (b \cdot ab)c = (b \cdot ba)c = (a \cdot bb)c \\ &= (c \cdot bb)a = (b \cdot bc)a = (b \cdot cb)a = (bc \cdot b)a = ca \\ \Rightarrow ac &= ca. \end{aligned}$$

(iii) To prove that H is left regular, let $ca = cb$. Then by left semisymmetry and Part (i)

$$\begin{aligned} ca &= cb \Rightarrow c(da \cdot d) = c(db \cdot d) \\ \Rightarrow da \cdot d &= db \cdot d \Rightarrow da = db. \end{aligned}$$

Thus H is left regular. To prove that H is right regular assume that $ac = bc$. By left semisymmetry, Part (i) and Corollary 5.3

$$\begin{aligned}(da \cdot d)c &= (db \cdot d)c \Rightarrow da \cdot d = db \cdot d \\ \Rightarrow d \cdot ad &= d \cdot bd \Rightarrow ad = bd.\end{aligned}$$

Thus H is right regular.

(iv) To prove that H is T_f^4, assume that $ab = cd$. Thus by left semisymmetry, Corollary 5.3, left invertive law and Part (i)

$$\begin{aligned}(da \cdot d)b &= (bc \cdot b)d \Rightarrow (d \cdot ad)b = (b \cdot cb)d \\ \Rightarrow (b \cdot ad)d &= (b \cdot cb)d \Rightarrow b \cdot ad = b \cdot cb \Rightarrow ad = cb.\end{aligned}$$

Thus H is T_f^4-AG-groupoid.

Hence the theorem is proved. □

Here, we provide a counterexample to show that anti-rectangular AG-groupoid is not T_b^4.

Example 5.5. *In Table* 7*, an anti-rectangular AG-groupoid of order* 8 *is given. Since* $2 \cdot 1 = 4 = 7 \cdot 8 \nRightarrow 8 \cdot 2 = 1 \cdot 7$ *so it is not* T_b^4*.*

·	1	2	3	4	5	6	7	8
1	1	3	4	2	5	8	6	7
2	4	2	1	3	6	7	5	8
3	2	4	3	1	8	5	7	6
4	3	1	2	4	7	6	8	5
5	5	7	6	8	1	2	4	3
6	7	5	8	6	3	4	2	1
7	8	6	7	5	2	1	3	4
8	6	8	5	7	4	3	1	2

Table 7

We further provide various other counterexamples to show that the converse of each part of Theorem 5.4 is not true.

Example 5.6. *A cancellative AG-groupoid of order* 4 *is given in Table* 8*, which is not anti-rectangular AG-groupoid as* $(1 \cdot 2)1 \neq 2$*. In Table* 9 *a Transitively commutative AG-groupoid of order* 5 *is given, as* $(1 \cdot 3)1 \neq 3$ *so it is not anti-rectangular.*

·	1	2	3	4
1	2	4	3	1
2	3	1	2	4
3	1	3	4	2
4	4	2	1	3

Table 8

·	1	2	3	4	5
1	2	2	1	1	1
2	2	2	2	2	2
3	1	2	3	3	3
4	1	2	3	3	3
5	1	2	3	3	3

Table 9

Example 5.7. *Table* 10 *represents a regular AG-groupoid of order* 4 *which is not anti-rectangular AG-groupoid as* $(2 \cdot 3)2 \neq 3$*. In Table* 11 *a* T_f^4*-AG-groupoid of order* 4 *is given, as* $(1 \cdot 3)1 \neq 3$ *thus it is not anti-rectangular.*

·	1	2	3	4
1	2	2	2	2
2	2	2	2	2
3	1	1	1	1
4	1	1	1	1

Table 10

·	1	2	3	4
1	2	2	1	2
2	2	2	1	2
3	4	4	4	4
4	1	1	2	1

Table 11

5.8. **Relations of Anti-rectangular AG-groupoids with Left and Right Distributive AG-groupoids.** First, we give some examples to show that the combination of anti-rectangular AG-groupoid with left distributive and right distributive AG-groupoid is non-associative. We further show that if an AG-groupoid is anti-rectangular then left distributivity and right distributivity coincide.

Example 5.9. *A non-associative anti-rectangular distributive AG-groupoid of order* 4 *is presented in Table* 12*.*

·	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Table 12

Theorem 5.10. *An anti-rectangular AG-groupoid is left distributive if and only if it is right distributive.*

Proof. Let H be a left distributive AG-groupoid and $a,\, b,\, c \in H$. Then by left invertive law, left semisymmetry, medial law and left distributive property

$$\begin{aligned} bc \cdot a &= ac \cdot b = (ac)(ab \cdot a) = (a \cdot ab)(ca) = (aa \cdot ab)(ca) \\ &= ((ab \cdot a)a)(ca) = ba \cdot ca \Rightarrow bc \cdot a = ba \cdot ca. \end{aligned}$$

Hence H is right distributive. Conversely, suppose H is right distributive, then by Lemma 5.1, left semisymmetry, Corollary 5.3, medial law and right distributive property

$$\begin{aligned} a \cdot bc &= c \cdot ba = (ac \cdot a)(ba) = (a \cdot ca)(ba) = (ab)(ca \cdot a) \\ &= (ab)(ca \cdot aa) = (ab)(a(a \cdot ca)) = (ab)(a(ac \cdot a)) = ab \cdot ac \\ \Rightarrow a \cdot bc &= ab \cdot ac. \end{aligned}$$

Hence H is left distributive. □

The following examples show that left or right distributivity do not guarantee of an anti-rectangular AG-groupoid.

Example 5.11. *Table* 13 *represents a left distributive AG-groupoid of order* 4 *which is neither anti-rectangular nor right distributive. While in Table* 14 *a right distributive AG-groupoid of order* 4 *is presented which is neither anti-rectangular nor left distributive.*

$\cdot$	1	2	3	4
1	2	1	1	1
2	1	2	2	2
3	1	2	2	2
4	1	2	2	2

Table 13

$\cdot$	1	2	3	4
1	2	2	2	2
2	3	3	3	3
3	3	3	3	3
4	2	2	2	2

Table 14

It was shown earlier that (left/right) distributive anti-rectangular AG-groupoid is non associative. Here we further give some examples to show that anti-rectangular anti-commutative AG-groupoid is also a non-associative structure. We present examples that there is no direct relation among anti-rectangular, anti-commutative and left distributive AG-groupoids, but if we combine anti-rectangular with anti-commutative AG-groupoid, then we obtain an AG-band. If we combine anti-rectangular AG-groupoid with the left distributive property, then the result is anti-commutative AG-band. Further, if we combine anti-rectangular AG-groupoid and anti-commutative AG-groupoid satisfying the condition $a^nb^n = b^na^n$ where n is even positive integer then it becomes unipotent AG-groupoid. We will give a counterexample to show that every unipotent AG-groupoid is not anti-rectangular AG-groupoid.

Example 5.12. *A non-associative anti-rectangular and anti-commutative AG-groupoid of order* 4 *is presented in Table* 15.

$\cdot$	1	2	3	4
1	1	3	1	4
2	4	1	3	2
3	3	2	4	1
4	1	4	2	3

Table 15

Example 5.13. *Table* 7 *of Example 5.5 is an anti-rectangular AG-groupoid of order* 8, *it is not* (i) *AG-band as* $5\cdot 5 \neq 5$, (ii) *left distributive as* $6(7\cdot 8) = 6\cdot 4 = 6$ *and* $(6\cdot 7)(6\cdot 8) = 2\cdot 1 = 4$, *thus* $6(7\cdot 8) \neq (6\cdot 7)(6\cdot 8)$, (iii) *anti-commutative as* $1\cdot 5 = 5\cdot 1 \nRightarrow 1 = 5$.

Example 5.14. *Left regular AG-groupoid of order* 4 *given in Table* 16, *which is neither* (i) *anti-rectangular as* $(3\cdot 4)3 \neq 4$, *nor* (ii) *AG-band, and nor* (iii) *anti-commutative as* $1\cdot 3 = 3\cdot 1 \nRightarrow 1 = 3$.

$\cdot$	1	2	3	4
1	2	2	2	2
2	2	2	2	2
3	2	2	1	1
4	2	2	1	1

Table 16

Example 5.15. *Anti-commutative AG-groupoid given in Table* 17 *is not* (i) *anti-rectangular as* $(1\cdot 2)1 \neq 2$, (ii) *AG-band as* $1\cdot 1 \neq 1$, (iii) *left distributive as* $1(2\cdot 3) = 1\cdot 3 = 1$ *and* $(1\cdot 2)(1\cdot 3) = 3\cdot 1 = 3,$ *thus* $1(2\cdot 3) \neq (1\cdot 2)(1\cdot 3)$.

$\cdot$	1	2	3	4
1	2	3	1	4
2	4	1	3	2
3	3	2	4	1
4	1	4	2	3

Table 17

Example 5.16. *Table* 18 *represents a unipotent AG-groupoid of order* 4 *which is not anti-rectangular AG-groupoid.*

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	d	c	a	b
d	c	d	b	a

Table 18

Theorem 5.17. *Every anti-rectangular AG-groupoid H is an AG-band if any of the following holds.*
(i) *H is anti-commutative,*
(ii) *H is left distributive.*

Proof. Let H be an anti-rectangular AG-groupoid and $a, b \in H$.

(i) Let $a \in H$. Then by Corollary 5.3 and anti-commutativity

$$aa \cdot a = a \cdot aa \quad \Rightarrow aa = a.$$

(ii) Let $a,\ b \in H$. Then by left semisymmetry, Corollary 5.3 and left distributive property

$$a \quad = \quad ba \cdot b = b \cdot ab = ba \cdot bb = (ba \cdot b)(ba \cdot b) = aa \Rightarrow a = aa.$$

Hence in each case H is AG-band and the theorem is proved. □

Theorem 5.18. *Every anti-rectangular left distributive AG-groupoid H is anti-commutative.*

Proof. Let $ab = ba$ for all $a,\ b \in H$. Then by left semisymmetry, Corollary (5.3), left distributive property, Theorem 5.17, Lemma (5.1), Corollary (5.3) and assumption

$$\begin{aligned} a &= ba \cdot b = b \cdot ab = ba \cdot bb = (b \cdot aa)(bb) = (a \cdot ab)(bb) \\ &= (a \cdot ba)(bb) = (ab \cdot a)(bb) = b \cdot bb = bb \cdot b = b. \end{aligned}$$

Equivalently H is anti-commutative. □

Theorem 5.19. *Anti-rectangular anti-commutative AG-groupoid H is unipotent if $a^n b^n = b^n a^n$ for all $a, b \in H$ where n is even positive integer.*

Proof. Suppose $a^n b^n = b^n a^n$ for all $a, b \in H$ and even positive integer n. Then by Lemma 5.1, Corollary 5.3 and left semisymmetry

$$\begin{aligned} a^n b^n &= b^n a^n \\ \Rightarrow (a^{n-2}a^2)(b^{n-2}b^2) &= (b^{n-2}b^2)(a^{n-2}a^2) \\ \Rightarrow (a^{n-2} \cdot aa)(b^{n-2} \cdot bb) &= (b^{n-2} \cdot bb)(a^{n-2} \cdot aa) \\ \Rightarrow (a \cdot aa^{n-2})(b \cdot bb^{n-2}) &= (b \cdot bb^{n-2})(a \cdot aa^{n-2}) \\ \Rightarrow (a \cdot a^{n-2}a)(b \cdot b^{n-2}b) &= (b \cdot b^{n-2}b)(a \cdot a^{n-2}a) \\ \Rightarrow (aa^{n-2} \cdot a)(bb^{n-2} \cdot b) &= (bb^{n-2} \cdot b)(aa^{n-2} \cdot a) \\ \Rightarrow a^{n-2}b^{n-2} &= b^{n-2}a^{n-2}. \end{aligned}$$

After repeating the same process for $\frac{1}{2}(n-2)$ times we get,

$$\begin{aligned} a^2 b^2 &= b^2 a^2 \text{ which by anti-commutativity gives} \\ a^2 &= b^2. \end{aligned}$$

Thus H is unipotent. □

5.20. **Relations of Anti-rectangular AG-groupoids with Semigroups.** Generally, AG-groupoid is non-associative but sometimes the combination of two different subclasses of AG-groupoids give rise to a semigroup. In this context, an anti-rectangular AG-groupoid has very close relation with a semigroup. Namely, when anti-rectangular AG-groupoid is combined with various other subclasses of AG-groupoid it becomes a semigroup. In the following theorem we list some of these subclasses.

Theorem 5.21. *An anti-rectangular AG-groupoid H is a semigroup if any of the following hold:*
(i) *H is paramedial,*
(ii) *H is LAD (left abelian distributive),*
(iii) *H is RAD (right abelian distributive),*
(iv) *H is left nuclear square,*
(v) *H is right nuclear square,*
(vi) *H is LC (left commutative),*
(vii) *H is RC (right commutative).*

Proof. Let H be an anti-rectangular AG-groupoid.

(i) Assume that H also satisfies the paramedial property. Then for any $a, b, c \in H$, by left semisymmetry, medial law, paramedial property, Theorem 5.1 and Corollary 5.3 we have,

$$\begin{aligned} ab \cdot c &= (ab)(ac \cdot a) = (a \cdot ac)(ba) = (c \cdot aa)(ba) \\ &= (a \cdot aa)(bc) = (aa \cdot a)(bc) = a \cdot bc. \end{aligned}$$

Hence H is semigroup.

(ii) Assume that H also satisfies the LAD property. Then for every $a,\ b,\ c \in H$, by LAD, left invertive law, left semisymmetry, medial law and Corollary 5.3 we have,

$$\begin{aligned} a \cdot bc &= ab \cdot ca = (ca \cdot b)a = (ca \cdot b)(ca \cdot c) = (ca \cdot ca)(bc) \\ &= ((ca \cdot c)(a \cdot ca))(bc) = ((ca \cdot c)(ac \cdot a))(bc) = ac \cdot bc \\ &= ab \cdot cc = (ab)(c(cc \cdot c)) = (ab)((c \cdot cc)(cc)) \\ &= (ab)((cc \cdot c)(cc)) = (ab)(c \cdot cc) = (ab)(cc \cdot c) = ab \cdot c. \end{aligned}$$

Hence H is semigroup.

(iii) Assume that H also satisfies the RAD property. Then for every $a,\ b,\ c \in H$, by RAD, left semisymmetry, Corollary 5.3, medial law and Corollary 5.3,

$$\begin{aligned} ab \cdot c &= ca \cdot bc = c(b \cdot ca) = (ac \cdot a)(b \cdot ca) = (a \cdot ca)(b \cdot ca) \\ &= (ab)(ca \cdot ca) = (ab)((ca \cdot c)(a \cdot ca)) = (ab)((ca \cdot c)(ac \cdot a)) \\ &= (ab)(ac) = (aa)(bc) = ((aa \cdot a)a)(bc) = ((a \cdot aa)(aa))(bc) \\ &= ((aa \cdot a)(aa))(bc) = (a \cdot aa)(bc) = (aa \cdot a)(bc) = a \cdot bc. \end{aligned}$$

Hence H is semigroup.

(iv) Assume that H also satisfies the left nuclear square property. Then for every $a,\ b,\ c \in H$, by left semisymmetry, medial law, left nuclear square, left invertive law and Corollary 5.3,

$$\begin{aligned} ab \cdot c &= (ab)(cc \cdot c) = (a \cdot cc)(bc) = ((aa \cdot a)(cc))(bc) = ((aa)(a \cdot cc))(bc) \\ &= ((aa)(c \cdot ca))(bc) = ((aa \cdot c)(ca))(bc) = ((ca \cdot c)(aa))(bc) \\ &= (a \cdot aa)(bc) = (aa \cdot a)(bc) = a \cdot bc. \end{aligned}$$

Hence H is semigroup.

(v) Assume that H also satisfies the right nuclear square property. Then for all $a,\ b,\ c \in H$, by left semisymmetry, medial law, Corollary 5.3, right nuclear square, left invertive law and Lemma 1we have,

$$\begin{aligned} a \cdot bc &= (aa \cdot a)(bc) = (a \cdot aa)(bc) = (ab)(aa \cdot c) = (ab)((aa)(cc \cdot c)) \\ &= (ab)((aa)(c \cdot cc)) = (ab)((aa \cdot c)(cc)) = (ab)((ca \cdot a)(cc)) \\ &= (ab)((ca)(a \cdot cc)) = (ab)((cc)(a \cdot ca)) = (ab)((cc)(ac \cdot a)) \\ &= (ab)(cc \cdot c) = ab \cdot c \Rightarrow a \cdot bc = ab \cdot c. \end{aligned}$$

Hence H is semigroup.

(vi) Assume that H also holds the LC property. Then for every $a,\ b,\ c \in H$, by assumption, left semisymmetry, left invertive law, Lemma 1 and medial law we have,

$$\begin{aligned} ab \cdot c &= ba \cdot c = ((ab \cdot a)a)c = (aa \cdot ab)c = (c \cdot ab)(aa) \\ &= (b \cdot ac)(aa) = (ba)(ac \cdot a) = (ba)(ca \cdot a) \\ &= a(ca \cdot ba) = a((ba \cdot a)c) = a((ab \cdot a)c) = a \cdot bc. \end{aligned}$$

Hence H is semigroup.

(vii) Assume H is also RC. Then for every $a,\ b,\ c \in H$, by assumption, left semisymmetry, Lemma 1, medial law and left invertive law we get,

$$\begin{aligned} a \cdot bc &= a \cdot cb = a(c(ab \cdot a)) = (ab \cdot a)(ca) = (ab \cdot c)(aa) \\ &= (cb \cdot a)(aa) = a(a(cb \cdot a)) = a(a(a \cdot cb)) = a(a(b \cdot ca)) \\ &= a(a(b \cdot ac)) = a(ac \cdot ba) = a(ac \cdot ab) = ab(ac \cdot a) = ab \cdot c. \end{aligned}$$

Hence H is a semigroup.

Hence in each case H satisfies the associative law and thus is a semigroup. □

We summarize the investigated relations of various AG-groupoids and other structures with the anti-rectangular AG-groupoids in the following table.

Various structures that contains anti-rectangular AG-groupoid as a subclass			
1.	Cancellative AG-groupoid	6.	Flexible AG-groupoid
2.	Self-dual AG-groupoid	7.	Quasigroup
3.	Regular AG-groupoid	8.	Locally associative AG-groupoid
4.	Transitively commutative AG-groupoid	9.	AG-3-band
5.	T_f^4-AG-gorupoid	10	T^3-AG-groupoid
AG-groupoid for which anti-rectangular becomes a smeigroup			
1.	Paramedial	3.	Left(right) commutative
2.	Left(right) abelian distributive	4.	Left(right) nuclear square

Relations of anti-rectangular with other structures

The concept of anti-rectangular AG-groupoids generalizes the class of an anti-rectangular AG-band as proved by P. V. Protić. However, it should be noted that not every anti-rectangular AG-groupoid is an anti-rectangular AG-band as shown by the following counterexample.

Example 5.22. *An anti-rectangular AG-groupoid of order* 8 *is presented in Table* 19 *which is not anti-rectangular AG-band.*

$\cdot$	1	2	3	4	5	6	7	8
1	3	4	1	6	2	8	5	7
2	5	2	6	1	4	7	3	8
3	1	7	3	5	8	2	6	4
4	2	5	8	7	3	1	4	6
5	7	1	4	2	6	5	8	3
6	4	3	7	8	5	6	2	1
7	8	6	2	4	1	3	7	5
8	6	8	5	3	7	4	1	2

Table 19

5.23. **Relations of Anti-rectangular AG-groupoids with Commutative Structures.** Here, we find the relation of anti-rectangular AG-groupoid with commutative structures. Basically anti-rectangular AG-groupoid is non-associative structure. Sometimes it is not possible for non-associative structure to become commutative without external conditions. Therefore, we put some extra conditions on anti-rectangular AG-groupoid to become a commutative structure. For instance if we take left identity in an anti-rectangular AG-groupoid it becomes an abelian group whose all elements are self inverse. We also show that anti-rectangular AG-groupoid H becomes a commutative semigroup if $a^n b^n = b^n a^n$ for every $a,\ b \in H$, where n is an odd integer greater than 1.

Theorem 5.24. *Every anti-rectangular AG-groupoid with left identity is an abelian group, in which each element is self inverse.*

Proof. Let H be an anti-rectangular AG-groupoid, $e,\ a\ \in H$, where e is the left identity and a is any element of H. Then by left semisymmetry and left invertive law we have,

$$ae = (ea \cdot e)e = ee \cdot ea = e \cdot a = a. \tag{5.1}$$

$$ab = ea \cdot b = ba \cdot e = ba. \tag{5.2}$$

$$ab \cdot c = cb \cdot a = bc \cdot a = a \cdot bc \Rightarrow ab \cdot c = a \cdot bc. \tag{5.3}$$

Thus H is associative by (5.3) and hence commutative monoid by (5.1) and (5.2). Finally, we show that each element of H is self inverse. For this let $b \in H$. Then by left semisymmetry,

$$bb = be \cdot b = e.$$

Thus $bb = e$. Hence b is its own inverse. □

Example 5.25. *Table* 20 *represents an anti-rectangular AG-groupoid* H *of order* 8 *with identity element* 1*, which is an abelian group having the property that each element is its own inverse.*

·	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	2	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

Table 20

Example 5.26. *An abelian group given in Table* 21 *which is not an anti-rectangular AG-groupoid as,* $(w \cdot 1)w \neq 1$.

$\cdot$	1	w	w^2
1	1	w	w^2
w	w	w^2	1
w^2	w^2	1	w

Table 21

Next, we establish a relation between abelian groups and anti-rectangular AG-groupoids.

Theorem 5.27. *An abelian group $(H, \cdot)$ is anti-rectangular AG-groupoid if all its elements are self inverse.*

Proof. Let H be an abelian group with identity e such that each of its element is self inverse and $x,\, y \in H$. Then,

$$ab \cdot c \quad = \quad a \cdot bc = bc \cdot a = cb \cdot a \Rightarrow ab \cdot c = cb \cdot a \tag{5.4}$$

and

$$ab \cdot a \quad = \quad ba \cdot a = b \cdot aa = b \cdot e = b \Rightarrow ab \cdot a = b. \tag{5.5}$$

Thus H satisfies the left invertive law by (5.4) and the left semisymmetry by (5.5). Hence H is anti-rectangular AG-groupoid. □

Example 5.28. *An abelian group whose each element is self inverse is the dihedral group of order 4, $D_2 = \{e,\, a,\, b,\, ab\colon\; a^2 = b^2 = (ab)^2 = e\}$ with the following Cayley's Table 22. It is easy to verify that D_2 is anti-rectangular AG-groupoid.*

$\cdot$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

Table 22

Theorem 5.29. *Anti-rectangular AG-groupoid H is commutative semigroup if $a^n b^n = b^n a^n$ for every $a,\, b \in H$, where n is odd positive integer greater than 1.*

Proof. Let $a^n b^n = b^n a^n$ for all $a,\, b \in H$ and n is odd integer greater than 1. Then by Lemma 5.1 and Corollary 5.3 and left semisymmetry

$$\begin{aligned}
a^n b^n &= b^n a^n \\
\Rightarrow (a^{n-2}a^2)(b^{n-2}b^2) &= (b^{n-2}b^2)(a^{n-2}a^2) \\
\Rightarrow (a^{n-2} \cdot aa)(b^{n-2} \cdot bb) &= (b^{n-2} \cdot bb)(a^{n-2} \cdot aa) \\
\Rightarrow (a \cdot aa^{n-2})(b \cdot bb^{n-2}) &= (b \cdot bb^{n-2})(a \cdot aa^{n-2}) \\
\Rightarrow (a \cdot a^{n-2}a)(b \cdot b^{n-2}b) &= (b \cdot b^{n-2}b)(a \cdot a^{n-2}a) \\
\Rightarrow (aa^{n-2} \cdot a)(bb^{n-2} \cdot b) &= (bb^{n-2} \cdot b)(aa^{n-2} \cdot a) \\
\Rightarrow a^{n-2}b^{n-2} &= b^{n-2}a^{n-2}.
\end{aligned}$$

After repeating the same process for $\frac{1}{2}(n-3)$ times, we get

$$\begin{aligned} a^3b^3 &= b^3a^3 \\ \Rightarrow (a^2a)(b^2b) &= (b^2b)(a^2a) \\ (aa \cdot a)(bb \cdot b) &= (bb \cdot b)(aa \cdot a) \\ \Rightarrow ab &= ba. \end{aligned}$$

Thus H is commutative. Since commutativity implies associativity in AG-groupoid [25], hence H is commutative semigroup. □

Proposition 5.30. *Let H be an anti-rectangular AG-groupoid and b be a fixed element in H. Then the set $(b)H = \{y \in H : \ by = y\}$ is empty or $b \in (b)H$.*

Proof. Suppose b is not idempotent and assume $y \in (b)H$. Then $by = y$, by using left semisymmetry, part (i) of Corollary 5.3 and Theorem 5.4

$$by = y \quad \Rightarrow by = by \cdot b \Rightarrow by = b \cdot yb \Rightarrow y = yb.$$

Thus $$by = y \quad \Rightarrow y = yb \Rightarrow by = yb.$$

Which show that b commutes with elements of $(b)H$.

Further $$by = y \Rightarrow yb = y \Rightarrow (by)b = y \Rightarrow (b \cdot by)b = by \cdot b$$
$$\Rightarrow \quad b \cdot by = by \Rightarrow y \cdot bb = by \Rightarrow y \cdot bb = yb \Rightarrow bb = b,$$

i.e. b is idempotent, a contradiction to our assumption, thus $(b)H = \phi$. On the other hand if b is idempotent, then by definition $b \in (b)H$. □

The converse of the above proposition is not valid. In Example 5.2 Table 5, let $H = \{1, 2, 3, 4, 5\}$. Then $b \in (b)H$ for all $b \in H$, but it is not anti-rectangular AG-groupoid.

Theorem 5.31. *Let H be an anti-rectangular AG-groupoid and b be a fixed idempotent element in H. Then $(b)H = \{y \in H : \ by = y\}$ is an abelian group, in which each element is self inverse with identity b.*

Proof. Clearly for any idempotent element b in H, $(b)H \neq \phi$ by Proposition 5.30. Let $x, y \in (b)H$. Then $x = bx$ & $y = by$. Thus by left semisymmetry, medial law and Lemma 5.1

$$xy \quad = \quad bx \cdot by = bb \cdot xy = b \cdot xy \Rightarrow xy = b \cdot xy \Rightarrow (b)H \text{ is closed.}$$

Again

$$xy \quad = \quad x \cdot by = y \cdot bx = yx \Rightarrow xy = yx \Rightarrow (b)H \text{ is commutative.}$$

Hence $(b)H$ is associative. Also $by = yb = y$ shows that b is the identity of $(b)H$. Finally, we show that each element of $(b)H$ is self inverse. Let $y \in (b)H$. Then by commutativity and left semisymmetry

$$yy \quad = \quad by \cdot y = yb \cdot y \Rightarrow yy = b.$$

Thus y is self inverse, hence the theorem follows. □

Example 5.32. $(H, \cdot)$ *of Table* 7 *in Example 5.5 is an anti-rectangular AG-groupoid,* $(1)H = \{1, 5\}$, $(2)H = \{2, 8\}$, $(3)H = \{3, 7\}$, $(4)H = \{4, 6\}$ *are abelian groups with identities* $1, 2, 3$ *and* 4 *respectively. While* $(5)H = (6)H = (7)H = (8)H = \phi$.

Definition 5.33. [4] *Let* H *be a non-empty set with binary operation ".". Then* H *is called quasigroup if the equations* $hx = k$ *and* $xh = k$ *each has exactly one solution for every* $h, k, x \in H$. *A quasigroup with identity element is called a loop, i.e. there exist an element* $e \in H$ *such that* $eh = he = h$ *for every* $h \in H$.

Theorem 5.34. *An anti-rectangular AG-groupoid is a quasigroup.*

Proof. Let K be an anti-rectangular AG-groupoid. Assume on the contrary, let the equation $kx = g$ has two solutions x_1, x_2. Then

$$kx_1 = g \tag{5.6}$$
$$kx_2 = g \tag{5.7}$$

From (5.6) and (5.7) we have $kx_1 = kx_2$, this by part (i) of Theorem 5.4 implies $x_1 = x_2$. This contradicts our assumption. Thus $kx = g$ has a unique solution. On similar way, let $xk = g$ has two solutions x_1, x_2. Then

$$x_1k = g \tag{5.8}$$
$$x_2k = g \tag{5.9}$$

From (5.8) and (5.9) we have $x_1k = x_2k$, this by part (i) of Theorem 5.4 implies $x_1 = x_2$. Thus $xk = g$ has also a unique solution. Hence the result follows. □

Theorem 5.35. *Let* K *be an anti-rectangular AG-groupoid. Then* $cK = Kc = K$ *for every* $c \in K$.

Proof. Let $h \in K$. Then by closure property and left semisymmetry

$$\begin{aligned} ch &\in K \Rightarrow ch \cdot c \in Kc \Rightarrow h \in Kc \\ \Rightarrow \bigcup_{k \in K} h &\subseteq \bigcup_{k \in K} hc \\ \Rightarrow K &\subseteq Kc \end{aligned} \tag{5.10}$$

Now, let

$$\begin{aligned} hc &\in K \Rightarrow \bigcup_{k \in K} hc \subseteq \bigcup_{k \in K} h \\ \Rightarrow Kc &\subseteq K \end{aligned} \tag{5.11}$$

$$\Rightarrow \quad Kc = K \quad (\text{by } 5.10) \text{ and } (5.11) \tag{5.12}$$

Again, let $g \in K$. Then by closure property, Corollary 5.3 and left semisymmetry

$$\begin{aligned} gc &\in K \Rightarrow c \cdot gc \in cK \\ \Rightarrow g &\in cK \Rightarrow \bigcup_{g \in K} g \subseteq \bigcup_{g \in K} cg \\ \Rightarrow K &\subseteq cK. \end{aligned} \tag{5.13}$$

And

$$\begin{aligned} cg &\in K \Rightarrow \bigcup_{g\in K} cg \subseteq \bigcup_{g\in K} g \\ cK &\subseteq K \end{aligned} \tag{5.14}$$

$$\Rightarrow \quad cK = K. \quad \text{(by } 5.13) \text{ and } (5.14) \tag{5.15}$$

Thus by (5.12) and (5.15) $cK = Kc = K$. □

Proposition 5.36. *Let K be an anti-rectangular AG-groupoid. Then for all $a,\, c \in K$ and any integer $m \geq 1$*

$$(ac)^m = \begin{cases} a^2c^2 & \text{if } m \text{ is even} \\ ac & \text{if } m \text{ is odd} \end{cases}$$

Proof. Let K be an anti-rectangular AG-groupoid. Then for all $a,\, c \in K$.
Case 1. Assume that $m \geq 1$ is odd. Then by Corollary 5.3, Lemma 5.1 and left semisymmetry we have,

$$\begin{aligned} (ac)^m &= a^mc^m = (a^{m-2}a^2)(c^{m-2}c^2) = (a^{m-2}\cdot aa)(c^{m-2}\cdot cc) \\ &= (a\cdot aa^{m-2})(c\cdot cc^{m-2}) = (a\cdot a^{m-2}a)(c\cdot c^{m-2}c) \\ &= (aa^{m-2}\cdot a)(cc^{m-2}\cdot c) = a^{m-2}c^{m-2}. \end{aligned}$$

Repeating the process $\frac{1}{2}(m-3)$ times we have

$$\begin{aligned} &= a^3c^3 = (a^2\cdot a)(c^2\cdot c) = (aa\cdot a)(cc\cdot c) = ac \\ \Rightarrow (ac)^m &= ac. \end{aligned}$$

Case 2. Assume that m is even. Then by Corollary 5.3, Lemma 5.1, Corollary 5.3 and left semisymmetry we get,

$$\begin{aligned} (ac)^m &= a^mc^m = (a^{m-2}a^2)(c^{m-2}c^2) = (a^{m-2}\cdot aa)(c^{m-2}\cdot cc) \\ &= (a\cdot aa^{m-2})(c\cdot cc^{m-2}) = (a\cdot a^{m-2}a)(c\cdot c^{m-2}c) \\ &= (aa^{m-2}\cdot a)(cc^{m-2}\cdot c) = a^{m-2}c^{m-2}. \end{aligned}$$

Repeating the process $\frac{1}{2}(m-2)$ times we have,

$$(ac)^m = a^2c^2.$$

□

5.37. **Anti-rectangular AG-groupoid and Cheban Loop.** Theorem 5.34 reveals that every anti-rectangular AG-groupoid is a quasigroup. It is also known that a quasigroup with neutral element is a loop. Here, we prove that an anti-rectangular AG-groupoid always satisfies the Cheban identity.

Theorem 5.38. *An anti-rectangular AG-groupoid is left and right Cheban.*

Proof. Let H be an anti-rectangular AG-groupoid and $a, b, c \in H$. Then by left semisymmetry, medial and left invertive laws, Corollary 5.3 and Lemma 5.1

$$\begin{aligned} a(ab \cdot c) &= (ba \cdot b)(ab \cdot c) = (ba \cdot ab)(bc) = (bc \cdot ab)(ba) = (ba \cdot cb)(ba) \\ &= (ba)(cb \cdot ba) = (ba)(a(b \cdot cb)) = (ba)(a(bc \cdot b)) = ba \cdot ac. \end{aligned}$$

Thus $a(ab \cdot c) = ba \cdot ac$. Similarly,

$$\begin{aligned} (c \cdot ba)a &= (c \cdot ba)(ba \cdot b) = (a \cdot bc)(ba \cdot b) = (a \cdot bc)(b \cdot ab) \\ &= (ab)(bc \cdot ab) = (ab \cdot bc)(ab) = (c(b \cdot ab))(ab) \\ &= (c(ba \cdot b))(ab) = ca \cdot ab. \end{aligned}$$

Thus $(c \cdot ba)a = ca \cdot ab$. Hence the result follows. □

Theorem 5.39. *Let h be an element of an anti-rectangular AG-groupoid H such that $ah = ha$ and $ch = hc$, where $a, c \in H$. Then a and c commute.*

Proof. By Lemma 5.1 and part (i) of Theorem 5.4

$$\begin{aligned} h(ac) &= c(ah) = c(ha) = a(hc) = a(ch) = h(ca) \\ \Rightarrow h(ac) &= h(ca) \Rightarrow ac = ca. \end{aligned}$$

Hence a and c commute. □

The converse of the above theorem is not valid. For instance consider Table 9 of Example 5.6 wherein $H = \{1, 2, 3, 4, 5\}$. Let $a = 1, c = 2$ and $h = 1$. Then $ac = ca = 2$ and $ah = ha = 2$ and $hc = ch = 2$. However H is not anti-rectangular.

Construction of an algebraic structures is always an important task. By defining new operators, construction of some specific groupoids, AG-groupoids and commutative structures from other known groupoids and AG-groupoids are given in [**?**, 2]. Here we construct permutable groupoids from anti-rectangular AG-groupoid.

Theorem 5.40. *Let $(K, \cdot)$ is an anti-rectangular AG-groupoid. Define $\circ$ on K as $a \circ b = ka \cdot b$, where k is a fixed element of K. Then $(K, \circ)$ is right permutable.*

Proof. Using left invertive law and Lemma 5.1

$$\begin{aligned} (a \circ b) \circ c &= (k(ka \cdot b))c = (c(ka \cdot b))k \\ &= (b(ka \cdot c))k = (k(ka \cdot c))b \\ \Rightarrow (a \circ b) \circ c &= (a \circ c) \circ b. \end{aligned}$$

Hence $(K, \circ)$ is right permutable. □

Theorem 5.40 does not guarantee that $(K, \circ)$ will be an AG-groupoid as verified below.

Example 5.41. *In Table 23, $(K, \cdot)$ is an anti-rectangular AG-groupoid of order 4. Using Theorem 5.40 and taking $k = 2$ as fixed, we get the Cayley's Table 24 of right permutable $(K, \circ)$, as $(2 \circ 1) \circ 3 \neq (3 \circ 1) \circ 2$, thus it is not an AG-groupoid.*

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Table 23

$\circ$	1	2	3	4
1	3	1	2	4
2	4	2	1	3
3	1	3	4	2
4	2	4	3	1

Table 24

6. IDEALS IN ANTI-RECTANGULAR AG-GROUPOIDS

Some researchers studied topological structures and proper ideals of anti-rectangular AG-groupoids. However, they did not provide a single example of it. We claim that this class is simple and have no proper ideals. We prove our claim in the next theorem.

Theorem 6.1. *An anti-rectangular AG-groupoid is simple.*

Proof. Let H be an anti-rectangular AG-groupoid. Assume on contrary that I be an ideal of H. Then by closure property, Corollary 5.3 and left semisymmetry

$$HI \subseteq I \quad \Rightarrow I(HI) \subseteq II \quad \Rightarrow (IH)I \subseteq II \subseteq I \Rightarrow H = (IH)I \subseteq I \Rightarrow H \subseteq I.$$

Since $I \subseteq H$. Hence $H = I$. Therefore H is simple. □

The converse of the above theorem is not true as every cancellative AG-groupoid is simple but is not anti-rectangular. We provide a counterexample to depict it.

Example 6.2. *Let* $H = \{0, 1, 2, 3, 4\}$. *Then cleary* $(H, \cdot)$ *is cancellative AG-groupoid. Since,* $0 = (1 \cdot 2) \cdot 1 \neq 2)$, *thus it is not an anti-rectangular AG-groupoid.*

$\cdot$	0	1	2	3	4
0	2	1	0	4	3
1	0	4	3	2	1
2	3	2	1	0	4
3	1	0	4	3	2
4	4	3	2	1	0

Table 25

7. CONCLUSION

We proved that an anti-rectangular AG-groupoid is a quasigroup and satisfies the properties of left and right Cheban. We demonstrated the relations of anti-rectangular AG-groupoid with different subclasses of AG-groupoid and found that it is simple. The number of anti-rectangular AG-groupoids of order 2 and 4 are respectively 1 and 2, while there exists no such AG-groupoid of order $3, 5, 6$, which indeed is an exceptional characteristic of anti-rectangular AG-groupoids. We also introduced anti-rectangular test for a finite AG-groupoid. Further, we investigated various properties for anti-rectangular AG-groupoids and proved that it is cancelllative, transitively commutative, regular, AG-3 band, local associative and flexible. We further proved that left and right distributivity coincide in anti-rectangular AG-groupoids. We provided various conditions under which an anti-rectangular AG-groupoid becomes a semigroup and a commutative group.

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