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Comprehensive notes on various effects of some operators of fractional-order derivatives to certain functions in the complex domains and some of related implications

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Abstract: The main objective of this comprehensive research note is to reacquaint certain necessary information with respect to various type operators designated by the fractional-order calculus in certain domains of the complex plane, then to reveal various significant effects specified by those fractional-order (type) operators for certain complex functions regular in the open unit disk, and also to center upon numerous possible implications of our main results, and to put emphasis on a number of special results of related implications in relation with several differential type analytic-geometric properties of those regular functions in the concerned open set.

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1. Introduction and Preliminary Information

Through the instrumentality of a simple literature survey, one can easily encounter unnumberable investigations associating with fractional-order calculus (that is, differentiation and integration of an arbitrary real (or complex) order). For its details, see [18, 21]. In addition, of course, there also exist their extensive applications containing different fields of science and technology. For some of those, see [23, 25, 29]. More specially, it can be proposed the earlier papers in relation with many different fields. For the interested readers, as examples, in [27], some results relating to "Fractional model and exact solutions of convection flow of an incompressible viscous fluid under the newtonian heating and mass diffusion" were obtained, in [28], general information together with various applications to univalent function theory associating with "Fractional calculus and their applications" were given, and, in [31], some results concerning "Analysis of the influences of parameters in the fractional second-grade fluid dynamics" were also presented. Naturally, the concept of the theory of the fractional calculus also deals with many fractional order operators. As we know, there also exist various operators of fractional-order type in the mathematical literature. Basically, a number of them are famous fractional-order derivative operators such as the Riemann-Liouville derivative and the Caputo fractional derivative cited as in [8] and [26]. In the same time, since it is possible that these fractional-order operators and their comprehensive implications play a different role for mathematical science, with the help of them, both new operators and some generalizations of those fractional operators can also be constituted. For certain different-type applications of Owa-Srivastava operator; in [11], some results pertaining to "A few applications to Janowski spiral-like functions" were determined; in [12], some general results regarding "Generalized Srivastava-Owa fractional operators" were obtained; in [20], several results concerning "K-uniformly convex type functions" were also derived. For the Tremblay Operator established by Fractional derivative(s) operator; in [6], some elementary results consisting of certain analytic functions in some classes specified by the related operator were obtained, and, in [12], various applications of both this operator together with some special applications of fractional calculus were also presented. In the same time, for some extra-different information and/or applications; in [2], a number of applications relating with "Differential subordination of higher-order derivatives of certain multivalent functions" were presented; in [4], the main information together with their properties concerning "Special functions and their applications" were pointed out; in [16], the main-lemma used for a great many applications involving analytic-geometric properties was looked over; in [18], the detailed information and their properties-different applications in connection with "Fractional Differential Equations" can also be reviewed. Indeed, this extensive investigation associates with a fractional-order (type) operator, which corresponds essentially to a special form of the classical Riemann-Liouville fractional derivative (or integral) of order μ (or $-\mu$) (cf., e.g., [15, 29]).

Moreover, for determining our main results consisting of various analytic-geometric properties of regular functions in the next section, we would like to point out some extra points in particular. Since this scientific research note will directly be appertaining to comprehensive applications of those operators of fractional order derivative to various complex functions regular in certain domains of the complex plane, there is a need to introduce extra necessary information, definitions, notations, notions, representations and some special elementary calculations. Additionally, at the end of this section, the motivation of our investigation will also be emphasized.

By the familiar notations $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, we represent the set of the natural numbers, the set of the real numbers and the set of the complex numbers, respectively.

Next, both in this section and also in the other sections of this research work, let the following conditions satisfy for the parameters γ, λ and μ :

$$0<\gamma\leq 1 \quad , \quad 0<\lambda\leq 1 \quad , \quad 0\leq\lambda-\gamma<1 \quad \text{and} \quad 0\leq\mu<1. \tag{1.1}$$

Let a function $\varphi(z)$ be regular in the following complex domain

$$\mathbb{U}:=\{w : w\in\mathbb{C} \text{ and } |w|<1\}$$

and also let it possess any complex-series expansion of the forms given by

$$\begin{aligned}\varphi(z) &:= \mathsf{L}_r^s(z)\\ &:= \chi_s z^s+\chi_{s+r}z^{s+r}+\chi_{s+r+1}z^{s+r+1}+\cdots \quad ,\end{aligned} \tag{1.2}$$

where

$$s\in\mathbb{N} \quad , \quad r\in\mathbb{N} \quad , \quad \chi_s\in\mathbb{C}-\{0\} \quad , \quad \chi_{s+r}\in\mathbb{C} \quad \text{and} \quad z\in\mathbb{U}. \tag{1.3}$$

As it has been indicated above, here, two comprehensive operators of fractional-order calculus are important to our research. Both operators will be considered for the functions which are regular in $\mathbb{U}$ and are of the series form in (1.2). As it was indicated before, the first is the well-known Fractional Derivative Operator, which corresponds essentially to the classical Riemann-Liouville fractional derivative (of order μ $(0\leq\mu<1)$). The other is one of certain types defined by the help of the operator of fractional-order derivative(s), which is usually called the Tremblay Operator.

For those, under the mentioned conditions in (1.1) and also for any function $\varphi(z)$ being the form (1.2), the mentioned operators of fractional-order derivatives, typically denoted by the notations:

$$\mathsf{T}^{\lambda,\gamma}[\varphi(z)] \quad \text{and} \quad \mathsf{D}_z^{\mu}[\varphi(z)],$$

are then defined by

$$\mathsf{T}^{\lambda,\gamma}[\varphi(z)]=\frac{\Gamma(\gamma)}{\Gamma(\lambda)}z^{1-\gamma}\mathsf{D}_z^{\lambda-\gamma}\left[z^{\lambda-1}\varphi(z)\right] \tag{1.4}$$

and

$$\mathsf{D}_z^{\mu}[\varphi(z)]=\frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z\frac{\varphi(\ell)}{(z-\ell)^{\mu}}\,d\ell, \tag{1.5}$$

where $z\in\mathbb{U}$ and the function $\varphi(z)$ is regular in any simply-connected region of the complex domain $\mathbb{U}$ comprising the origin and the multiplicity of $(z-\ell)^{-\mu}$ is insulated by making use of $log(z-\ell)$ when $z-\ell>0$. As well, both here and in parallel with this paper, the notation Γ (just above) denotes the Gamma Euler function.

Clearly, for any regular function being of the form (1.2), the operator in the definition (1.5) has been used for new operator in the definition (1.4), which relates to one of the special forms given by

$$z^{\rho}\,\mathsf{D}_z^{\rho}[\varphi(z)] \quad \left(0\leq\rho<1\right). \tag{1.6}$$

Specially, as it has been noted before, for the regular function $\varphi(z)$, those fractional-order type operators given in (1.4)-(1.6) are generally encountered, respectively, as the

Tremblay Operator, the Fractional Derivative Operator (of order ρ $(0 \leq \rho < 1)$) and the Srivastava-Owa Operator in the mathematical literature (cf., e.g., [6], [12], [14], [20], [22] and [30]).

As two simple examples, after certain special-elementary calculation, by means of the mentioned operators given by the definitions in (1.4) and (1.5), for an elementary-regular function being of the form:

$$\zeta(z) = z^{\kappa} \quad (\kappa > -1),$$

the following results:

$$\mathsf{D}_z^{\rho}[\zeta(z)] = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\rho+1)} z^{\kappa-\rho} \quad \big(0 \leq \rho < 1\big) \tag{1.7}$$

and

$$\mathsf{T}^{\lambda,\gamma}[\zeta(z)] = \frac{\Gamma(\gamma)\Gamma(\kappa+\lambda)}{\Gamma(\lambda)\Gamma(\kappa+\gamma)} z^{\kappa} \tag{1.8}$$

can easily be composed under the restricted conditions of the related parameters given in (1.1).

In the light of the examples above, as some of the more special results of the associated operators of fractional-order calculus, it immediately follows from (1.7) and (1.8) that the following more special relationships given by

$$\mathsf{D}_z^{0}[\zeta(z)] = \zeta(z), \tag{1.9}$$

$$\lim_{\rho \to 1^-} \mathsf{D}_z^{\rho}[\zeta(z)] = \zeta'(z), \tag{1.10}$$

$$\lim_{\rho \to 0^+} z^{\rho}\, \mathsf{D}_z^{\rho}[\zeta(z)] = \zeta(z), \tag{1.11}$$

$$\lim_{\rho \to 1^-} z^{\rho}\, \mathsf{D}_z^{\rho}[\zeta(z)] = z\zeta'(z) \tag{1.12}$$

and

$$\mathsf{T}^{\tau,\tau}[\zeta(z)] = \zeta(z), \tag{1.13}$$

where $0 \leq \rho < 1$, $0 < \tau \leq 1$ and $z \in \mathbb{U}$.

As the last definition of this section, under the restricted conditions given by (1.1), and for all values of the parameter j $(j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ and also for any regular function being of the form in (1.2), by taking into account the following-equivalent expression:

$$\frac{d^j}{dz^j}\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big) \equiv \Big(\mathsf{T}_r^{s}\big[\varphi(z)\big]\Big)^{(j)} \quad \big(z \in \mathbb{U}\big) \tag{1.14}$$

for the jth derivative of the concerned operator in (1.4) with respect to the complex variable z, the scope of this research will also be detailed.

Most especially, by combining of the basic results in (1.7) and (1.8) and, also, in view of the definition in (1.14) with $j \leq s$ $\big(j \in \mathbb{N}_0; s \in \mathbb{N}\big)$, for any function with complex variable:

$$\varphi(z) := \mathsf{L}_r^{s}(z) \quad \big(r, s \in \mathbb{N}\big),$$

which is regular in $\mathbb{U}$, the extensive assertions can easily be determined, which are designed by

$$\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big] = \chi_s \,\mathrm{I}_s^{\lambda,\gamma} z^s + \sum_{k=s+r}^{\infty} \chi_k \,\mathrm{I}_k^{\lambda,\gamma} z^k \tag{1.15}$$

and

$$\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(j)} = \chi_s \,\mathrm{J}_{s,j}^{\lambda,\gamma} z^{s-j} + \sum_{k=s+r}^{\infty} \chi_k \,\mathrm{J}_{k,j}^{\lambda,\gamma} z^{k-j}, \tag{1.16}$$

where

$$\mathrm{I}_s^{\lambda,\gamma} = \frac{\Gamma(\gamma)\Gamma(s+\lambda)}{\Gamma(\lambda)\Gamma(s+\gamma)} \tag{1.17}$$

and

$$\mathrm{J}_{s,j}^{\lambda,\gamma} = \frac{s!}{(s-j)!}\,\mathrm{I}_s^{\lambda,\gamma} \quad (j \leq s) \tag{1.18}$$

for all natural numbers $s \in \mathbb{N}$ and $j \in \mathbb{N}_0$.

We especially point out here that, as a motivation of this investigation, by the earlier paper given by [14], an extensive result, which relates with both the general form in (1.14) (a long with (1.15) and (1.16)) and a number of its argument properties, was added variety to the published literature. By this scientific investigation, some of extra comprehensive results are also planned to establish for certain regular functions being of the series-expansion form given by (1.2). Specially, in the light of the extensive information given by the definitions between (1.15)-(1.18), for those new results which will be presented in the third section, we also note that all main results will be established by considering only suitable conditions for all natural numbers with $r \leq s$ there.

2. Related Lemma and Main Results

In this second section, a necessary lemma, the main results of our research and various implications and suggestions regarding those main results will also be presented.

As a very common proof method, the following auxiliary theorem, which is Lemma 2.1, will be required to prove our fundamental results. For the detail of its proof, the main references, given in [17] and [20], can be examined.

Lemma 2.1. *Let $\Xi(v)$ be a function with complex variable that is regular in U and is of the form in (1.2). For any point $v_0 \in \mathbb{U}$, if*

$$\big|\Xi(v_0)\big| = max\Big\{\,\big|\Xi(v)\big| \,:\, |v| \leq |v_0| \,\left(v \in \mathbb{U}\right)\Big\}, \tag{2.1}$$

then there exists a number ω such that

$$v_0\,\Xi'(v_0) = \omega\,\Xi(v_0), \tag{2.2}$$

where $\omega \in \mathbb{R}$ with $\omega \geq s$ and $s \in \mathbb{N}$.

We can now begin by establishing our basic results together with some of their comprehensive implications, which will be consisted of real part, imaginary part and modulus of various complex-type expressions created by the help of the information between (1.15)-(1.18).

Theorem 2.2. *Under the conditions presented as in the information in (1.1), (1.3), (1.4) and (1.5), let the parameters v, s, α and β supply the designated conditions given by*

$$v \geq r \ , \ r \in \mathbb{N} \ , \ s \in \mathbb{N} \ , \ 0 < \beta \leq 1 \ \textit{and} \ 0 \leq \alpha < 2\pi. \tag{2.3}$$

For every z in the domain $\mathbb{U}$ and for any regular function $\varphi(z)$ having the form (1.2), if the following inequality:

$$\Re\left\{z\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(1+s)}\right\} \neq v\pi\beta Cos(\alpha) \tag{2.4}$$

is satisfied, then the inequality:

$$\left|\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(s)} - \chi_s J^{\lambda,\gamma}_{s,s}\right| < \beta \tag{2.5}$$

is also satisfied, where the notation $J^{\lambda,\gamma}_{s,s}$ denotes the same notation designated as in (1.18) together with (1.17).

Proof. Let the complex function $\varphi(z)$ possess the form in (1.2). Under the conditions in (1.1), (1.3) and (2.2), and also by means of the information presented in (1.15)-(1.18), let us consider a complex function $\mathcal{V}(z)$ in the implicit form defined by

$$\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(s)} = \chi_s \mathrm{J}^{\lambda,\gamma}_{s,s} + b\mathcal{V}(z), \tag{2.6}$$

where $\mathrm{J}^{\lambda,\gamma}_{s,s}$ is defined by (1.18) (and (1.17)) and, of course, $0 < b \leq 1$, $\chi_s \in \mathbb{C} - \{0\}$ and $z \in \mathbb{U}$.

When taking into consideration the elementary information between (1.14) and (1.16), it can be easily seen that the function $\mathcal{V}(z)$ defined in (2.6) is a regular function $\mathbb{U}$ and has any form in (1.2) when $s := r$ $(r \in \mathbb{N})$. In other words, it can be considered for the desired proof of Theorem 2.2. Then, from (2.6) together with (1.14), it instantly follows that

$$z\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(1+s)} = bz\mathcal{V}'(z) \quad \big(0 < b \leq 1; s \in \mathbb{N}; z \in \mathbb{U}\big). \tag{2.7}$$

We systematically claim that $|\mathcal{V}(z)| < 1$ is true for all z in $\mathbb{U}$. However, if not, then, in conformity with the assertion in (2.1) (of the related lemma), there exists a point z_0 in $\mathbb{U}$ such that

$$\big|\mathcal{V}(z_0)\big| = \max\Big\{\big|\mathcal{V}(z)\big| : |z| \leq |z_0| \ \big(z \in \mathbb{U}\big)\Big\} = 1,$$

which also causes that

$$\mathcal{V}(z_0) = e^{i\theta} \quad \big(0 \leq \theta < 2\pi; z_0 \in \mathbb{U}\big).$$

Therefore, the expression in (2.1) also presents us

$$z_0\mathcal{V}'(z_0) = c\mathcal{V}(z_0) \quad \big(c \geq r; r \in \mathbb{N}\big).$$

Therefore, as a result of the information above, for all $c \geq r$ $(r \in \mathbb{N})$, by selecting z as z_0 and considering the real part of the related expression determined in (2.7), it can easily be arrived at the assertions given by

$$
\begin{aligned}
\Re\left\{ z\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(1+s)}\bigg|_{z:=z_0}\right\} &= \Re\left\{ z_0\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\big(z_0\big)\Big)^{(1+s)}\right\} \\
&= \Re\Big\{ bz_0\mathcal{V}'\big(z_0\big)\Big\} \\
&= \Re\Big\{ bc\mathcal{V}\big(z_0\big)\Big\} \\
&= \Re\Big\{ cbe^{i\theta}\Big\} \\
&= cb\mathrm{Cos}(\theta),
\end{aligned}
\tag{2.8}
$$

where $c \geq r$, $r \in \mathsf{N}$, $0 < b \leq 1$ and $0 \leq \theta < 2\pi$. Clearly, the result determined by (2.8) is a contradiction with the result presented by (2.4) when the parameters c, θ and b above are chosen as

$$c := v \quad , \quad \theta := \alpha \quad \text{and} \quad b := \beta,$$

respectively. This enunciates us that there is no a point $z_0 \in \mathbb{U}$ satisfying the condition $|\mathcal{V}(z_0)| = 1$. Moreover, this also requires to be $|\mathcal{V}(z)| < 1$ in $\mathbb{U}$. Therefore, the expression, stated in (2.6), immediately follows that the inequality:

$$\left|\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(s)} - \chi_s \mathrm{J}_{s,s}^{\lambda,\gamma}\right| = b\big|\mathcal{V}(z)\big| < b, \tag{2.9}$$

where $0 < b \leq 1$, $\chi_s \in \mathbb{C} - \{0\}$ and $z \in \mathbb{U}$. Consequently, when taking $b := \beta$ in (2.9), the desired proof also completes.

As it can clearly be seen in Theorem 2.3 (just below), an extra comprehensive theorem consisting of four statements can also be created (or presented) as the second main result of this investigation.

Theorem 2.3. *Under the conditions particularized by (1.1), (1.3), (1.4), (1.5) and (2.3), for every $z \in \mathbb{U}$ and for all regular functions like $\varphi(z)$ in the form (1.2), if any one of the inequalities created by*

$$\Im\left\{ z\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(1+s)}\right\} \neq v\pi\beta \mathit{Sin}(\alpha), \tag{2.10}$$

$$\left| z\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(1+s)}\right| \neq v\pi\beta, \tag{2.11}$$

$$\left|\Re\left\{ z\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(1+s)}\right\}\right| \neq v\pi\beta\big|\mathit{Cos}(\alpha)\big| \tag{2.12}$$

and

$$\left|\Im\left\{z\Big(\mathsf{T}^{\lambda,\gamma}\left[\varphi(z)\right]\Big)^{(1+s)}\right\}\right| \neq v\pi\beta\big|\mathit{Sin}(\alpha)\big| \tag{2.13}$$

is provided, then the inequality in (2.5) is also provided.

Proof. The proof of Theorem 2.3 can easily be composed by the interested researchers. For each one of those statements there, it will be sufficient to consider the specific definition constituted by (2.6) in the proof of Theorem 2.2 and then follow all similar-mentioned steps in order there, namely, in the proof of the related theorem. Therefore, the details of the pending proof are omitted here.

3. Related Implications and Recommendations

In this third section, we would like also to present certain comprehensive information for our readers as various special implications together with recommendations regarding the previous section.

As emphasized in the first section, it can easily be observed that our main results may also be presented to researchers numerous comprehensive special results when the admissible values of each one of the parameters taken part in the main operators and their possible implications are selected. Moreover, a number of concerned relationships associating with those specific results have been presented in (1.7)-(1.13). When those accentual results are considered for the extensive assertions in (1.14)-(1.18), the diversity of the main results is easily seen. More specifically, we point out that all related-special results, which can also be obtained from our main results, will be related to either the *multivalent* functions *or* the *normalized* functions, which are regular in the domain $\mathbb{U}$. As it is well known, both the *multivalently* regular functions (in $\mathbb{U}$) and the *normalized* regular functions (in $\mathbb{U}$) are functions of special importance in the theory of univalent functions. Regarding this special field in the complex function theory, the fundamental works in [5], [9], [10] and [24], and also the earlier papers in [2], [3] and [20] can be proposed to related researchers.

In addition, as a result of a comprehensive research, when the relevant parameters in the expression obtained in (1.15) and/or the coefficients of the relevant series are selected appropriately, the special-extensive relations in both the expressions in (1.15) and (1.16) and the *hypergeometric* functions are also revealed. For those special series expansions (and related comprehensive implications), one can also focus on the works in [1], [4], [7] and [14].

For the first sample implication, in view of the definition of the regular function in (1.2), let $\varphi(z) := \mathrm{L}_1^3(z)$. Then, the complex function $\varphi(z)$ is of the series expansion given by

$$\varphi(z) := \chi_3 z^3 + \chi_4 z^4 + \chi_5 z^5 + \cdots \quad \big(z \in \mathbb{U}\big). \tag{2.14}$$

In the same breath, with the help of the special information presented in (1.15) and (1.16), the elementary results given by

$$\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big] = \chi_3\,\frac{\lambda(1+\lambda)(2+\lambda)}{\gamma(1+\gamma)(2+\gamma)}z^3 + \chi_4\,\frac{\lambda(1+\lambda)(2+\lambda)(3+\lambda)}{\gamma(1+\gamma)(2+\gamma)(3+\gamma)}z^4+\cdots \tag{2.15}$$

and

$$\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(3)} = \frac{3!}{0!}\chi_3\,\frac{\lambda(1+\lambda)(2+\lambda)}{\gamma(1+\gamma)(2+\gamma)} + \frac{4!}{1!}\chi_4\,\frac{\lambda(1+\lambda)(2+\lambda)(3+\lambda)}{\gamma(1+\gamma)(2+\gamma)(3+\gamma)}z+\cdots + \frac{5!}{2!}\chi_5\,\frac{\lambda(1+\lambda)(2+\lambda)(3+\lambda)}{\gamma(1+\gamma)(2+\gamma)(3+\gamma)}z^2+\cdots \tag{2.16}$$

are also obtained, where $z\in\mathbb{U}$.

For the second sample implication, by setting $\lambda:=\kappa$ and $\gamma:=\kappa$ in (2.15) and (2.16) and considering the special property in (1.9), the important relation between the operators in (1.4) and (1.5):

$$\mathsf{T}^{\kappa,\kappa}\big[\varphi(z)\big] \equiv z^{1-\kappa}\,\mathsf{D}_z^0\big[z^{\kappa-1}\varphi(z)\big] \equiv \varphi(z)$$

can easily be received, where $0<\kappa\leq 1$ and $z\in\mathbb{U}$. Thence, for the function $\varphi(z)$ having the form in (2.14), this special relation (just above) can also be presented by the following special results:

$$\mathsf{T}^{\kappa,\kappa}\big[\varphi(z)\big] = \varphi(z) = \chi_3 z^3+\chi_4 z^4+\chi_5 z^5+\cdots \tag{2.17}$$

and

$$\Big(\mathsf{T}^{\kappa,\kappa}\big[\varphi(z)\big]\Big)^{(3)} = \varphi'''(z) = \frac{3!}{0!}\chi_3+\frac{4!}{1!}\chi_4\,z+\frac{5!}{2!}\chi_5\,z^2+\cdots \tag{2.18}$$

are easily received, where $z\in\mathbb{U}$.

Now, in the light of the specific information presented in (2.14)-(2.17), we want to constitute only two implications of our main results for our researchers. The others are also omitted here. Nevertheless, as some extra-possible examples, one can concentrate on the earlier-specific results in the paper given in [14] in the references.

The detail of the first implication is hidden in Theorem 2.2. Therefore, it can easily be composed by the combining of the special information given in (2.15) and (2.16) for the 3-valently regular function $\varphi(z)$ possessing the form in (2.14). In short, for its expression, it is enough to take $s:=3$ and $r:=1$ in Theorem 2.2. That is also contained in Corollary 3.4 (just below).

Corollary 3.4. For the admissible values of the parameters v, λ, γ, β and α considered by the conditions in (1.1) and (2.3), and also for the regular function $\varphi(z)$ given by (2.14), the following proposition:

$$\Re\left\{z\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(4)}\right\} \neq v\pi\mathrm{Cos}\big(\alpha\big)$$
$$\Longrightarrow \left|\Big(\mathsf{T}^{\lambda,\gamma}\big[\varphi(z)\big]\Big)^{(3)} - 6\chi_3\,\frac{\lambda(1+\lambda)(2+\lambda)}{\gamma(1+\gamma)(2+\gamma)}\right| < \beta$$

is asserted, where $v \geq 1$, $z \in \mathbb{U}$ and, of course, $\chi_3 \neq 0$.

The detail of the second implication is hidden in Theorem 2.2 (*or*, in Corollary 3.4). Hence, for its statement, it is enough to use the special results given by (2.17) and (2.18). Shortly, it can easily be created by taking $\lambda := \kappa$ and $\gamma := \kappa$ $(0 < \kappa \leq 1)$ in Corollary 3.4 (*or*, equivalently, taking $s := 3$, $r := 1$, $\lambda := \kappa$ and $\gamma := \kappa$ $(0 < \kappa \leq 1)$ in Theorem 2.2) (cf., e.g., [5, 9, 24]), which is designated by the following corollary.

Corollary 3.5. For the admissible values of the parameters v, β and α considered by the conditions in (2.3), and also for the regular function $\varphi(z)$ given by (2.14), the following propositions:

$$\Re\Big(z\varphi''''(z)\Big) \neq v\pi\mathrm{Cos}\big(\alpha\big)$$
$$\Longrightarrow \big|\varphi'''(z) - 6\chi_3\big| < \beta$$
$$\Longrightarrow 6\Re\big(\chi_3\big) - \beta \leq \Re\big\{\varphi'''(z)\big\} \leq 6\Re\big(\chi_3\big) + \beta$$

are satisfied, where $v \geq 1$ and $z \in \mathbb{U}$.

4. Discussion and Conclusion

Since this research focuses on some effects of the fractional derivative operator to certain analytic functions, in the first part, some literature information about that operator and some of its applications has primarily been presented. In the second section of this scientific research, which contains quite comprehensive results, certain necessary basic information has then been reminded for concerned researchers, some operators of fractional-order calculus and a number of their possible effects (associating with various differential type analytic-geometric properties of those special functions with complex variable) have also been determined. In the third section, two main theorems specified by the related-fractional-order (type) operators and some of those implications have next been presented as some examples. By considering the detailed explanations accentuated in both the second section and the third section, both the revealing of other possible-specific implications (or special consequences) and suitable formations of some possible examples are also brought to the attention of the relevant researchers. Nevertheless, as some elementary examples, of course, in light of the specific-detailed information accentuated in this research and under special conditions consisting of certain reasonable-appropriate values of the parameters presented in our main results, by considering the following elementar-complex functions:

$$\begin{aligned}\varphi(z) &= z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots + \frac{1}{n!}z^n + \cdots \\ &\equiv e^z - 1 \quad \big(z \in \mathbb{U}\big)\end{aligned}$$

and

$$\begin{aligned}\varphi(z) &= z + z^2 + z^3 + \cdots + z^n + \cdots \\ &\equiv \frac{z}{1-z} \quad \big(z \in \mathbb{U}\big),\end{aligned}$$

one can refocus on those indicated-special investigations (or examinations). As a final word, of course, performing elementary operations with complex (- valued) functions is not an easy task. However, with the help of various computer programs and under appropriate-logical conditions, various more specific results, which can be derived by our results, in 2D and 3D dimensions can also be generated. Since our research is a theoretical research, the accentuated studies (or investigations) are only brought to the attention of the concerned researchers.

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