Beliefs, Processes and Difficulties Associated with Mathematical Problem Solving of Grade 9 Students

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ABSTRACT

Problem solving is a core component of mathematics education around the world. However, students face a number of challenges when engaged in mathematical problem solving. This paper examines the beliefs, processes and difficulties associated with mathematical problem solving of Grade 9 students. Consistent with the constructivist notions, we framed the study within Mayer's work who approached problem solving as a process that is largely influenced by problem representation and problem solution. We conducted semi-structured interviews with 12 Grade 9 students and further engaged them in solving five different word problems. The findings revealed that they struggle with solving mathematical word problems due to five major reasons. These include: making sense of the problem statement, conceptual understanding, contextualization, visualization of the problem, and critical thinking and reasoning. We conclude the paper by discussing relevant implications for mathematics education in Pakistan.

Introduction

One of the critical components of mathematical activities at all levels of education involves problem solving. Yet, students from across the world, consider it a difficult and boring task, and have low self-efficacy beliefs (Jan & Rodrigues, 2012; Stylianides & Stylianides, 2014; Tambychika & Meerah, 2010). The situation is particularly challenging in Pakistan, where there is a little focus on conceptual understanding, in-depth learning, and reflective thinking in mathematics education (Ali, 2011). Consequently, students struggle as they attempt to solve mathematical word problems. This paper presents an in-depth analysis of the beliefs, processes and difficulties associated with mathematical problem solving of Grade 9 students.

Mathematical problem solving is a cognitively challenging task that involves a mathematical problem/situation for which a solution is not readily accessible. The problem solver makes a number of attempts before finding a path that leads to the correct solution of the problem (Stylianides & Stylianides, 2014). Overall, problem solving in mathematics does not necessarily require a pre-determined or mechanical procedure; rather it needs more effort, contextualization of the problem within everyday life, visual imagery and may also involve higher risks of failure (Garderen, 2006; Krawec, 2010).

Problem solving has been conceptualized in different ways with a shift in its focal point (Ellis & III, 2005; Schoenfeld, 1992). Earlier models of mathematical problem solving emphasize mechanical and decontextualized procedures with a strong emphasis on basic skills and systematic steps to solve riddles or mathematical equations (e.g., Bransford & Stein, 1984; Newell & Simon, 1972). These approaches do not integrate problem solving within curriculum and/or everyday life experiences. Moreover, it is assumed that an abstract and rather detached understanding of problem solving skills are readily transferable to novel situations.

However, subsequent approaches to mathematical problem solving accentuate other essential yet critical aspects. For example, the influence of cognitive perspective in mathematical problem solving emphasize complex mental processes including cognition, metacognition, and information processing (Wheeler & Regian, 1999). The constructivist views, largely influenced by Piaget's work (1932, 1971), emphasize that individuals construct the interpretations of the world around them through their own interpretive frameworks. Consistently, mathematical problem solving is viewed as a process where the individual is an active participant who is responsible for the construction of problem solution through interaction of new and previous knowledge (Bautista, 2012; Mayer, 1983, 1985, 1998; Sepeng & Madzorera, 2014). Whereas, the social-cognitive view argues that learning is a social act that cannot be separated from the context. Therefore, it is important to embed mathematical problems into well-defined contexts and situations (Reusser & Stebler, 1997; Stylianides & Stylianides, 2014). Similarly, the situational or narrative perspective situates human intentions in the context of action (Bruner, 1985). Proponents of this view foreground the context in learning and argue that it is important to examine how do learning outcomes develop in context (Turner & Nolen, 2015). This situative mode of thought develops a deep mathematical thinking by helping the individual to focus on the context of the problem in order to understand the situation and obtain a correct mathematical solution (Rosales, Vicente, Chamoso, Muñez, & Orrantia, 2012). The socio-cultural approach suggests that diverse social, cultural, and linguistic backgrounds influence the ways in which students interpret mathematical activities and interactions in multi-cultural classrooms (Sepeng & Webb, 2012). In all, there is a shift in mathematical problem solving approaches from a procedural to cognitive, constructivist, situational and/or contextual mode of thinking.

Consistent with the constructivist perspective, Mayer (1983) viewed problem solving as a complex, multiple-step cognitive process which requires one to associate previous experiences to the problem at hand and further act upon the solution. He argued that a problem must be paraphrased, comprehensively understood, and then visually integrated into a theoretically correct and complete schematic structure in order to reach the solution. He identified *problem representation* and *problem solution* as two major processes involved in solving word problems (Mayer, 1985).

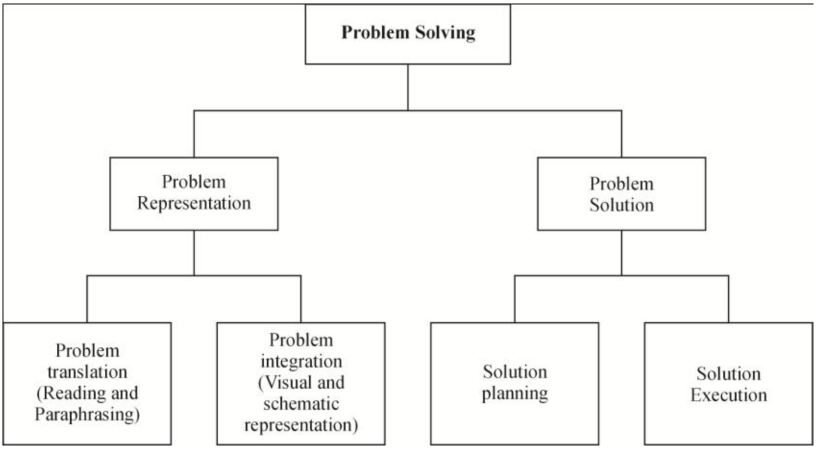

Figure 1. Mathematical problem solving - A conceptual framework based on Mayer's work (1985).

Problem representation is composed of two sub-stages including *problem translation* and *problem integration*; whereas problem solution comprises sub-stages including *solution planning* and *solution execution*. The two sub-stages of problem representation are essential precursors to a conceptually correct and comprehensive interpretation of the problem, which is strongly linked to successful problem-solving (Krawec, 2010). We used Mayer's approach to problem solving as a conceptual framework to situate this study (see Figure 1).

Problem translation is a necessary prerequisite for developing a correct and comprehensive understanding of the problem. It involves reading and paraphrasing the problem (often called re-telling) so that it is transformed into an understandable form for the problem solver (see Fig. 1.1). Reading and paraphrasing are the most critical steps for eventual success in problem-solving (Krawec, 2010; Montague, 2003). Any errors made at this point strongly influence the subsequent processes and solution accuracy. In mathematics research, paraphrasing does not only require students to re-word the text into a familiar form, but it also requires a critical analysis of the text for relevance. For example, students need to differentiate relevant information from irrelevant information, identify relationships, manipulate and process information based on relevance in order to understand the problem (Passolunghi, Marzocchi, & Fiorillo, 2005).

Problem integration, on the other hand, involves using a visual representation to comprehend the problem and its structure, and further interpret the relationships (Mayer, 1985). The research on visualization and its possible relationship with problem solving in mathematics identifies visual representations along a continuum ranging from pictorial to schematic. A pictorial representation of the problem emphasize pictures and images, whereas a schematic representation accentuates relationships and structures of the given problem (Krawec, 2010). While it is important that visual representations of the problem should include correct factual/numerical information, it should also include a schematic component that highlights spatial relationships between objects. The accuracy of representation along a schematic continuum, therefore, is critical to successfully solve word problems.

It is not a surprise that a majority of research in mathematical problem solving has focused on the latter part of the Mayer's model, that is *problem solution*, as demonstrated in Figure 1, with little or no emphasis on the first part (Krawec, 2010). This trend is similar to the prevalent norms at schools, where students' mathematical problem solving ability and performance are judged by evaluating the accuracy of problem solution rather than emphasizing problem representation and/or processing. Generally, mathematics teaching focuses on traditional practices and lacks complexity and reasoning (Ali, 2011; Jan & Rodrigues, 2012; Sullivan, 2011) which further explains why there is an over-emphasis on the accuracy of problem solution in schools. Moreover, although spatial and visual representations have received greater attention in mathematics research (Ahmada, Tarmizia, & Nawawi, 2010; Garderen, 2006; Moore & Carlson, 2012), there is a decided lack of research on different aspects of paraphrasing related to mathematical word problems (Krawec, 2010). The purpose of this study is to focus on the problem translation phase of Mayer's model (1985) while examining the beliefs, processes and difficulties associated with mathematical problem solving of Grade 9 students. Consistently, we frame the study within these research questions:

1. What are Grade 9 students' beliefs about mathematical problem solving?
2. How do Grade 9 students solve mathematical word problems?
3. What are the difficulties associated with mathematical problem solving of Grade 9 students?

Students require a range of skills and strategies from computational proficiency to sense making and contextualization within real world in order to solve mathematical word problems (Reusser & Stebler, 1997; Sepeng & Sigola, 2013; Stylianides & Stylianides, 2014). Successful problem solving in mathematics does not only require a command over basic mathematical skills such as number fact, arithmetic, information, language and visual-spatial competencies (Tambychika, Meerahb, & Azizb, 2010). Rather, it also involves the use of real-world knowledge, adaptive reasoning, attending to the contextual information, paraphrasing and analyzing the text, and exercising cognitive, metacognitive and motivational skills (Krawec, 2010; Mayer, 1998). We argue that a lack of these skills will inhibit students' ability to successfully solve mathematical problems.

A number of studies have documented the evidence for students' failure in mathematical problem solving. Evidence claims that students lack cognitive, mathematical, adaptive reasoning, critical thinking, visualization and interpretation skills that are critical for successful problem solving in mathematics (Ali, 2011; Krawec, 2010; Sepeng &

Sigola, 2013; Tambychika, et al., 2010). It is also noted that students do not understand mathematical vocabulary and make deeply ingrained conceptual errors which limit their ability to solve the word problems correctly. Students tend to exclude realistic considerations as they respond to the real-world problems (Reusser & Stebler, 1997). This tendency is attributed to the stereo type instructional practices in classrooms. Additionally, language plays an important role in students' ability to solve mathematical problems. Students tend to be more successful in solving word problems in their native language (Jan & Rodrigues, 2012). In contrast, they experience difficulties in comprehending word problems in a language they are not proficient at.

The National Curriculum for Mathematics (Grades I-XII) (Government of Pakistan, 2006) emphasizes that mathematics teaching in schools should enable students to apply their knowledge skillfully. It comprises five broad standards including numbers and operations, algebra, measurements and geometry, information handling, and reasoning and logical thinking. However, these standards are absent in mathematics curriculum and instruction, through which students can operationalize mathematical knowledge for work demands and living in a technological society (Ali, 2011). Instead, procedural knowledge is predominantly emphasized. For example, Standard-5, which includes reasoning and logical thinking, in the National Curriculum for Mathematics (Government of Pakistan, 2006), expects students to be able to "use patterns, known facts, properties and relationships to analyze mathematical situations" (p. 7). However, it does not focus on the factual relationship between mathematical operations and real world situation, which is critical to analyze mathematical situations. Similarly, there is rather an ambiguous emphasis on how to contextualize mathematical knowledge and proficiency within real life experiences. The benchmarks for 'reasoning and logical thinking' at Grade 9-10 level expect students to "choose appropriate strategy to solve mathematical problems" and "show step by step deduction in solving a problem" (p. 7). We argue that there is a need to move beyond the systematic, 'step by step' acquisition of procedural and declarative knowledge in the national mathematics curriculum. As an example, we can refer to the National Council of Teachers of Mathematics in the United States (National Council of Teachers of Mathematics, 2000). The council urged the need to incorporate the processes of problem solving, reasoning and proof, connections, communications and representations into the mathematics curriculum at every grade level.

Methodology

Constructivist ideas are central to mathematics education and research around the world (O'Shea & Leavy, 2013; Wood, Cobb, & Yackel, 1995). In general, three perspectives on constructivism have provided foundations for most of the scholarly work related to mathematical problem solving. These include: radical, social and emergent constructivism. Radical constructivism identifies learning as a process that mainly constitutes a series of cognitive reorganization of the individual's experiences and construction of knowledge (Glasersfeld, 1995a, 1995b; Piaget, 1971). Whereas, social constructivism argues that learning is a social accomplishment that is largely driven by social interactions (Bauersfeld, 1992; Vygotsky, 1978). The emergent view of constructivism emphasizes that learning and knowledge are individually constructed and socially mediated. Here, the individual and social domains of learning complement each other (Wood, et al., 1995).

The research questions of this study focus on the beliefs, processes and difficulties associated with mathematical problem solving of Grade 9 students and are anchored in the radical constructivist perspective (Glasersfeld, 1995a, 1995b; Piaget, 1971). Consistently, we believe that the participants of our study are proactive constructors, and their beliefs and knowledge are shaped up by their previous experiences. It is, therefore, important that we examine how do students engage in mathematical problem solving? How do they arrange given information as they solve the problems? Do they use previous knowledge? If yes, how? Do they construct new ways of interpreting and solving mathematical problems?

A qualitative research design allowed us to develop thick descriptions of the underlying beliefs and processes of the participants of the study who served as important sources of knowledge (Goldin, 2000; Hesse-Biber & Leavy, 2011; Steffe & Kieren, 1994). We attended to the *what* and *how* questions from their perspectives and ways of working, which allowed us to prioritize and project meaning(s) that they constructed based on their own experiences (Yee & Bostic, 2014). Data were collected through semi-structured interviews and students' written responses to five word problems administered to them after the interview.

Based on a convenience sampling technique (Patton, 2002), the participation in the study was completely voluntary. A total of 12 Grade

9 students (6 male, 6 female) from two public schools in Lahore, Pakistan agreed to participate in the study. We sought formal permissions from relevant gatekeepers such as principal and teachers before approaching students and inviting them to participate in the study. The students were clearly explained the purpose and procedures of the study and further made clear the right to withdraw from the study at any time without any penalty. The average age of students who agreed to participate was 14 years.

Semi-structured interview were conducted with the participants after school hours. The interviewer explored their views about mathematical problem solving; and further probed them to explain the processes they engage in, strategies they use, and challenges they encounter. Each interview lasted for about 10-15 minutes, was conducted in their native language (Urdu), digitally recorded and transcribed verbatim.

Each participant was requested to solve five word problems on the sheets provided to them for the purpose of data gathering. Time was not a consideration as they worked with the problems. The word problems were specifically focused on the standards provided by the National Curriculum for Mathematics - Grade I-XII (Government of Pakistan, 2006). These include, for example, numbers and operations, information handling, reasoning and logical thinking. The problems were presented in English language which is a foreign language for the participants, but is the language of instruction used in the Mathematics curriculum and text books used in this subject area.

An examination of word problems in the Mathematics textbook for Grade 9 (Government of Punjab, 2016) suggests that most of the problems appear as artificial and disguised exercises of formal mathematical operations and formulae with minimal focus on everyday knowledge and experience as shown in Figure 2. We argue that four (no. 2, 3, 4, and 5) out of five problems have no apparent synchronization with everyday life. We further argue that these problems offer no opportunities for students to engage with practical and useable mathematics; instead there is an over emphasis on methods (matrix inversion method), rules (Crammer's rule), and procedures/operations (dimensions of the rectangle, angles of the triangle, acute angles of the right triangle). Only, the last problem (no. 6) is contextualized within a realistic context to engage students with the intent of using given context as a stimulus for applying mathematical knowledge.

Solve the following word problems by using

(i) matrix inversion method (ii) Crammer's rule.

2. The length of a rectangle is 4 times its width. The perimeter of the rectangle is 150 cm. Find the dimensions of the rectangle.

3. Two sides of a rectangle differ by 3.5 cm. Find the dimensions of the rectangle if its perimeter is 67 cm.

4. The third angle of an isosceles triangle is 16° less than the sum of the two equal angles. Find three angles of the triangle.

5. One acute angle of a right triangle is 12° more than twice the other acute angle. Find the acute angles of the right triangle.

6. Two cars that are 600 km apart are moving towards each other. Their speeds differ by 6 km per hour and the cars are 123 km apart after 4 1/2 hours. Find the speed of each car.

Source: Government of Punjab, 2016, p. 28.

Figure 2. Examples of word problems included in Mathematics textbook for Grade 9 students

For the purpose of data collection, three word problems were presented to students with similarities in their design, structure, language and difficulty level to the ones present in the text book but, with a context. For example, word problem 1 (WP1), stated: 'Every week, Ayesha swims three more than six times the number of laps Sana swims. If Ayesha swims 45 laps, how many laps does Sana swim?" However, we also included two problems that emphasize mathematical operations and procedures, for example, WP4 stated: "The numbers 2, 3, 5 and *x* have an average equal to 4. What is *x*?"

As we analyzed the interview transcripts, we started with a belief that students may have different and unique views about how they approach problem solving in mathematics. Later, we examined if there were any emerging patterns that may give some shape to the overall views of the participants. The process began with an open coding approach. Here, we assigned labels to different segments of data, such as, 'incorrect translation.' Later, we organized the set of codes and tried to generate links between initial codes through axial coding. For example, we grouped together 'knowledge gaps,' 'incorrect translation' and 'lack of procedural/mathematical fluency'. Lastly, we highlighted dominant themes or patterns, such as, 'making sense of the word problems' by employing selective coding. This approach helped us to condense the list of themes (Richards, 2009).

While we explored data to portray students' thinking through the analysis of their verbal products (i.e., interview transcript), we also examined their actions (i.e., written responses to the given word problems) to make comparisons and highlight contradictions. This examination was framed within the first part of Mayer's model (1985) that strongly emphasizes problem representation (i.e., problem translation and problem integration/visualization) as a critical step towards solving mathematical problem. Consequently, we examine how do students translate and paraphrase word problems? How do they organize information? How do they devise new ways of thinking? What are the strategies used and errors made at this stage? How does this stage inform the problem solving process? This strategy allowed us to triangulate evidence from different sources in order to present a detailed account of Grade 9 students' beliefs, processes and difficulties associated with mathematical problem solving.

Results and Discussions

The results are offered through five main themes which emerged as we analyze data to present the bigger picture related to beliefs, processes and difficulties associated with mathematical problem solving of Grade 9 students. These include: making sense of the word problems, conceptual understanding, contextualization, visualization of the problem, and critical thinking and reasoning. Grounded in constructivist notions (Glasersfeld, 1995b), we frame the results in the first part of Mayer's model (1985) that emphasizes problem representation (i.e., translation and integration/visualization) as a critical stage of problem solving process.

Making sense of the word problem

The data (both verbal and written) suggest that Grade 9 students struggle with making sense of mathematical word problems in the first instance. They face several difficulties in translating a given problem, organizing information and transforming it into a mathematical representation (Mayer, 1985). These include, for example, difficulties in reading, understanding, and correctly recording the language and factual information given in the problem statement, decoding meaning of words, and lacking in mathematical and procedural fluency. Consequently, errors made at this stage interfere with their ability to devise a plan for problem solution.

Many students reported during the interviews that they find it difficult to make sense of the word problems. For example, one of the boys, B3, mentioned: "It is difficult to understand the [problem] statement [while solving mathematical problems], whereas in *mathematics* [emphasis added], we are given a simple equation." Similarly, a girl G2, said, "The hardest part [in problem solving] is the *statement*. If the statement is easy, then we understand [the problem], if it is difficult, then we don't." Although students generally find it hard to understand the problem statement, especially those who come from English as a foreign language context (Ahmada, et al., 2010; Jan & Rodrigues, 2012; Sepeng & Sigola, 2013); it was interesting to note that participants of the study tend to view problem solving as a separate entity from real mathematics. For example, B3 differentiated problem solving from mathematics and said, ". . . in *mathematics* [emphasis added], we are given a simple equation." This shows that students do not recognize problem solving as an integral component of useful and practical mathematics in everyday life. Instead, they view mathematical/ computational proficiency in a rather limited, isolated and disconnected fashion by excluding problem solving.

Students' written responses to the word problems further demonstrate that they struggled with making sense of the word problems because of their inability to read, understand and correctly record and organize the factual and relational information given in the problem statement. For example, most of the students could not understand the linguistic component ("three more than six times") involved in WP1 (Every week, Ayesha swims three more than six times the number of laps Sana swims. If Ayesha swims 45 laps, how many laps does Sana swim?). Consequently, B1 and B2, translated and represented it as "6x" rather than "3 + 6x" (see Figure. 3), which shows their inability to comprehend the language of the given word problem.

The inability to comprehend the language might be due to the nature of wording used in WP1 (Sepeng & Sigola, 2013). However, the way WP1 is coded in terms of language is similar to the word problems included in Grade 9 mathematics text book. For example, "4 times its width", "16° less than the sum of the two equal angles", and "12° more than twice the other acute angle" (see Figure 2) (Government of Punjab, 2016, p. 28).

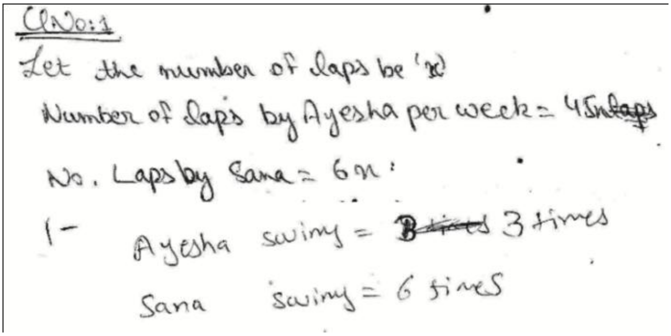

Figure 3. Students' inability to comprehend the language.

This explicit failure of students to decode the language of the text inhibits their ability to arrive at a correct numerical representation and problem solution. Students' failure can be attributed to the prevalent instructional practices that often provide them with the numerical statement of a problem without having them understand how and in what way problem is stated (Jan & Rodrigues, 2012). This was evident, when one of the participants G4, was prompted to think about the strategies she generally uses to solve mathematical word problems. She replied: "I prefer those [ways] that the teacher has taught us. I do not attempt to solve [problems] in my own [way] because I don't know if it's going to be right or not. . ." Her comments indicate that she is reluctant to devise and engage in new ways of thinking about mathematical problems.

It became evident through data that students did not know the meaning of mathematical terms such as *average*. For example, in her response to WP4 (The numbers 2, 3, 5 and x have an average equal to 4. What is x?), one of the girls, G5, wrote the formula for *average* as: "Sum of number / Total numbers". This implies that Grade 9 students have not yet mastered or developed fluency of basic mathematical procedures such as *average*. Mathematical fluency is regarded as the capacity to accurately, efficiently, easily and appropriately perform mathematical procedures. Research suggests that it is critical to develop mathematical fluency for all students in order to become effective problem solvers (Sullivan, 2011).

While there is no standard recipe to solve a mathematical word problem, the solution accuracy depends largely on the ability of students to correctly translate and visualize the problem into a model of inquiry

(Bautista, 2012; Mayer, 1985). The data from the study suggest that Grade 9 students struggle with making sense of mathematical word problems in the first place. They find it difficult to develop a comprehensive understanding of the word problems presented to them. They also faced problems in translating the word problem into an appropriate and understandable form. This is mainly due to the difficulties in reading, understanding, correctly recording the language and factual information given in the problem statement, decoding meaning of words, and lacking in mathematical and procedural fluency.

Conceptual understanding

The participants of the study believed that mathematical problem solving is merely about symbols, numbers, equations, procedures and rules. For example, G1, mentioned that mathematical problem solving is, "a calculation in which we use all [basic operations] like addition, multiplication." Similarly, B5 stated: "Sometimes we have a formula, sometimes we have an equation, we have to solve them [in mathematical problem solving]". Another student, B1, asserted: "*Whatever* the problem is, *whatever* the question is, you have to solve it by following a mathematical rule." Students' comments suggest that mathematical problem solving is all about procedural knowledge or rule-driven computation.

This mindset is also reflected in their written responses, as they attempt to solve every word problem by formulating a mathematical equation. However, a lack of mathematical fluency as noted earlier, hinders their ability to construct correct conceptual connections and representations. Consequently, they make conceptual errors, for example, B4 represented WP4 as shown in Figure 4.

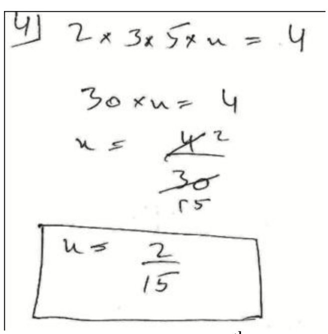

Figure 4. Conceptual errors in 9th grade mathematics.

B4 appears to lack in the conceptual understanding of mathematical concepts, procedures, and relationships (Kilpatrick, Swafford, & Findell, 2001; Sullivan, 2011). He seems to have a rather fractured knowledge of basic mathematical operations. His response to WP4 as shown in Figure 4. reflects a limited instrumental understanding of how to perform a mathematical task/operation. Although he formed an equation to solve the given problem, but he could not relate to the concept of *average* correctly. He did not understand that the term *average* implies addition not multiplication. Thus, he failed to demonstrate a relational understanding of mathematical concepts by teasing out an accurate relational representation that actually reflects WP4.

It is important to note that students did not only experience difficulties due to wrong interpretation of the language, but also grappled with representing the problem in a correct mathematical format. For example WP2 stated: Ali fills 60 cups of water from a 500-ounce water cooler. If there are 20 ounces left in the cooler when Ali has finished, and each cup contains the same amount of water, how much water is in each cup? While one of the participant B5, properly understood and interpreted the language of the problem, he could not represent the problem in the correct numerical/mathematical format, leading himself to a wrong solution (see Figure. 5).

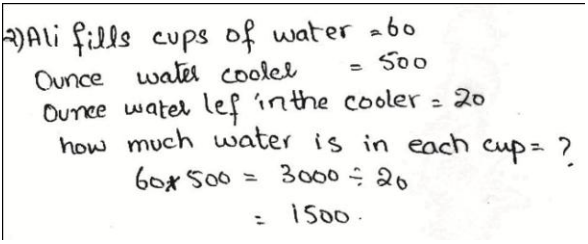

Figure 5. Incorrect mathematical representation.

Students' inability to transform the written text into accurate mathematical symbols and operations suggests that they mainly engage in computations without a conceptual understanding of what kind of mathematical operation is needed (instrumental understanding) and why is it needed (relational understanding). This tendency has already been reported in literature (Jan & Rodrigues, 2012). Thus, students' ability to translate and represent mathematical problems is not necessarily restricted by their linguistic capacities, but the capacity to conceptualize and represent the problem in correct mathematical terms also limits their problem solving skills.

In all, participants of the study were unable to correctly represent the given problems due to conceptual difficulties. These included both instrumental as well as relational failures on their part.

Contextualization

Students' comments (e.g., G1, B5, B1) presented in the above section further imply that a stringent emphasis on mathematical operations such as rules, methods, and formulae limit their ability to attend to the contextual information present in the problem statement. They strongly believe that, "*Whatever* the problem is, *whatever* the question is, you have to solve it by following a mathematical rule" (B1). Similarly they approach mathematical problem solving as, ". . . whatever is written in the question, solving it with a formula [is called mathematical problem solving]" (G6). Consistent with their beliefs, they "think about which formula to use, which method to use, [as they attempt] to solve word problems" (G4).

While the participants have a strong tendency to solve word problems by setting up mathematical equations and applying formulae, they rarely translated a real situation into proper mathematical form. This often resulted in working out with wrong solutions. These incorrect translations by students were largely attributed to their neglect of contextual information or considerations. A context is a realistic situation in which the mathematical problem is embedded to illustrate the potential application of mathematical concepts. It further provides opportunities for real world mathematical thinking and reasoning (Sullivan, 2011). For example, WP3 stated: Tanya has Rs. 25 in her bank. She adds one-fifth of the money she received for her birthday to the bank, bringing the total to Rs. 41. How much money did Tanya receive for her birthday? Figure 6 illustrates the written response to this problem by one of the boys (B6).

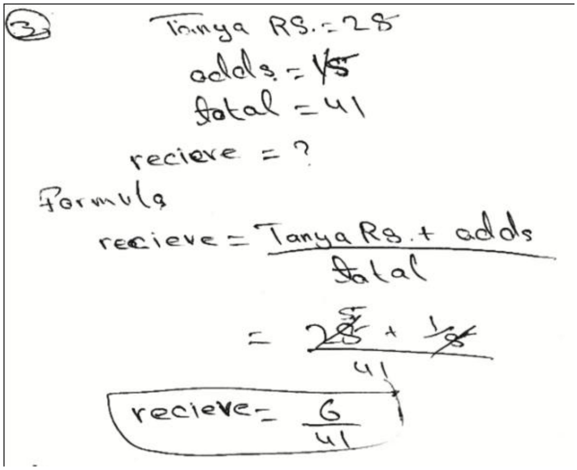

Figure 6. Ignoring the implied real-world situation.

Contextualization from a radical constructivist perspective focuses on understandings that are constructed by students based on their present and prior experiences, and communicated through language or written work (Yee & Bostic, 2014). It is evident from Figure 6 that B6 did not pay attention to the contextual information present in the problem statement. Instead of thinking about the implied real-world situation (i.e., situational context), he preferred to use or rather devise a formula, that did not accurately translate the word problem. He did not consider that money added to the bank was "one-fifth of the money Tanya received for her birthday." By ignoring this contextual information, he began the problem solving process with an incorrect translation that the money deposited to the bank was "1/5" only. He calculated the *total* money that Tanya received for her birthday to be "6/41" which equates to about Rs. 0.15. However, he did not realize that if *one-fifth* of the total money that Tanya received for her birthday equals to Rs. 16, then how come the *total* money that she received for her birthday turned out to be less than Rs.1? Thus, he was unable to interact with the situational context of the problem that carries information about the materials, actions, and environment. Similarly, his personal construction of the concepts of the situation was incorrect. This finding is consistent with the claims that students struggle with the notions of everyday life with regards to mathematical problem solving (Reusser & Stebler, 1997; Sepeng & Sigola, 2013).

Although students frequently solve word problems while ignoring contextual information, there were incidents when they successfully reached the solution without attending to the context in their written responses. For example, although G5 correctly solved WP2, her problem solving procedure did not account for the given context as illustrated in Figure. 7. She plainly organized numbers from the problem's text without a meaningful direction in her calculations. She did not even attempt to connect her mathematical reasoning with the given context and plainly wrote her answer as "8 = x," which gives us no indication that each cup of water contains 8 ounces.

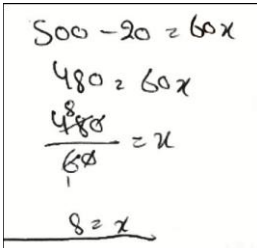

Figure 7. Organizing numbers/ information without clear directions.

The above discussion implies that students readily solve word problems by focusing on mathematical procedures with little or no consideration of contextual information. There is a strong evidence that students tend to exclude real world knowledge and context when solving word problems. This is because they could not connect school mathematics with everyday mathematics (Sepeng & Sigola, 2013). However, it is highly relevant to consider the fact that most often students are trained to deal with word problems without considering contextual information in mathematics instruction (Ali, 2011; Jan & Rodrigues, 2012). This tendency might be due to the way word problems are presented in mathematics text books (e.g., Government of Punjab, 2016), and problem solving is visualized in mathematics curriculum (Government of Pakistan, 2006).

Visualization of the problem

None of the participants from the study talked about and/or attempted to use meaningful, concrete, and schematic images to represent word problems. Although they were prompted to discuss different strategies employed by them when they get stuck while solving a mathematical problem during the interviews, none of them mentioned visualization as a way of understanding and representing the problem. It appeared to us as if they were not even familiar that they could derive meaning from the problem by visualizing and connecting different pieces of information. The use of schematic images has been regarded as a powerful problem representation tool that leads to successful problem solving in mathematics. Researchers argue that effective problem solvers generate meaningful and schematic visual representations that result in accurate solution of the problems (Garderen, 2006; Krawec, 2010; Mayer, 1985).

It is important to note, however, that we did not prompt students to draw diagrams or visual images as we administered word problems during data collection activity. The fact that none of the students attempted to draw a single visual image further supports our assumption about their limited beliefs and exposure to effective processes/strategies involved in mathematical problem solving. Researchers argue that students' preferences depend on what the teachers have taught them in their classrooms (Ahmada, et al., 2010; Garderen, 2006). Since students' approaches towards learning mathematical concepts are shaped up by their classroom experiences, it appears to us that the instructional practices and materials (e.g., textbook) within the context of this study, do not support the practice of representing word problems in the form of a concrete image.

Critical thinking and reasoning

The participants of the study reported that they face difficulties in solving word problems that are *new* to them, which implies that they are reluctant to experiment with new ways of thinking and constructing knowledge. For example, G4 mentioned: "If the question [word problem] is unseen, then it looks difficult. Sometimes, the examples in the books which are different from the exercise questions or stated methods, appear difficult [to us]." Consistently, they were not able to adapt their mathematical knowledge and understanding to different and novel situations. As a result, they tend to apply such formulae or equations to word problems that are either wrong or not required. In line with this finding, our results further show that students lack critical thinking and reasoning skills. For example, 11 out of 12 students gave partially correct response to WP5 that states: "Which number(s) is (are) equal to its (their) square?" However, all respondents identified "1" as the only correct answer. They did not attend to the clue present in the problem statement, "number(s) is (are) equal to its (their)" that urged them to think about more than one correct answer (square of zero equals to zero). In another incident, G1 solved WP1 as follows:

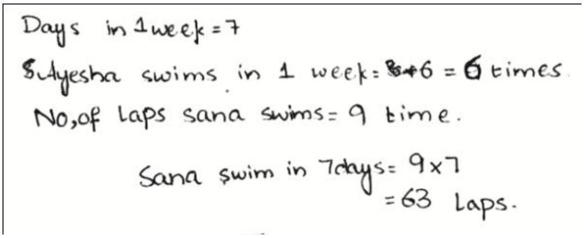

Figure 8. Lack of critical thinking and reasoning in mathematics

Figure 8 is a representation of G1's thinking as she attempted to solve WP1. However, there are critical flaws in the representation of the problem. G1 did not understand, organize and interpret the information given in WP1 correctly. Her written response indicates that she did not make use of this information in a logical manner to arrive at the conclusion. For example, she notes the number of laps that Ayesha swims as '6 times', which was actually 'three more than six times the number of laps Sana swims.' Although she initially represented her thought as '3+6' but then scribbled over it and settled with '6 times.' This shows that she did not ignore the information, rather she lacked critical thinking to figure out that the scheme 'three more than six times the number of laps Sana swims' needs to be broken down into two parts: 3 and *6x*. Moreover, she represented the number of laps that Sana swims as '9 time' and included it in the given set of information. This shows her inability to manipulate and process information based on logic and relevance. Instead, she simply used numbers without thinking about how to organize and relate these numbers to solve the problem.

This shows that students lack the capacity for logical thought, reflection, explanation and justification which is considered critical for mathematical problem solving (Sullivan, 2011). Instead, they focus on rule-driven computations or procedures during mathematical problem solving.

Conclusions

Framed within Mayer's work (1985), we embraced constructivist ideas to examine the beliefs, processes, and difficulties associated with mathematical problem solving of Grade 9 students. In doing so, we particularly focused on how do they translate, paraphrase, relate and organize the given information; what are the strategies they use and errors they make, and how does problem representation stage inform the problem solving process.

The study demonstrates that Grade 9 students view and engage in mathematical problem solving as a mechanical, systematic and often decontextualized or abstract set of skills with a strict focus on mathematical rules, equations and other procedures. A major consideration raised by our results is that students tend to view problem solving as a separate entity from mathematics. Consequently, they were not able to make sense of the word problems presented to them. Nor could they translate, conceptualize and reconstruct these problems into an appropriate model of inquiry which is critical for successful problem solving (Glasersfeld, 1995b; Mayer, 1985). Lack of or rather an absence of students' tendency to visually represent word problems also interfere with their problem solving processes (Krawec, 2010). This inability is often related to the instructional focus on non-visual proofs and numerical explanations along with a lack of value some teachers place on visual representations (Ali, 2011; Jan & Rodrigues, 2012). These findings draw our attention to the question of how to integrate visualization as a part of teaching and learning mathematical problem solving in our classrooms. Consistently, students should be taught to paraphrase, translate, organize, relate and visually represent word problem as their ultimate strategies to begin the problem solving process. While students benefit from such instruction, an added emphasis on how do they make sense of word problems and conceptual understanding may increase its effectiveness along with improved students' performance.

The study further highlights that students keep contexts separate from the logic of mathematical problem solving, and as a result, do not account for the contextual or situational aspects of the problem. However, they may not be able to take their knowledge and understanding of mathematics outside the walls of their classrooms unless curriculum, instructional practices and materials emphasize synchronization of mathematical knowledge within everyday life experiences (Sepeng & Sigola, 2013). An examination of national curriculum standards and mathematics textbook highlight stereotyped nature of word problems with an over-emphasis on rule-driven computations. Conversely, there is a need for better problems and better contexts (Reusser & Stebler, 1997). Instead of blaming students for their mathematical behaviour, we need to understand that it is shaped up by our curricular and instructional practices. In turn, we need to embed mathematical structures into richer description of story-like real world situations and train teachers to encourage students to contextualize mathematical problems. This may help students to activate and synchronize mathematical knowledge and understanding with their everyday experiences.

Our results further highlight that students do not engage in critical thinking and reasoning behaviours when solving mathematical word problems. It is often emphasized that students need to develop and master mathematical skills in order to engage in effective problem solving. However, the ultimate goal of mathematical problem solving is to learn and apply critical thinking and reasoning skills in everyday life (Krawec, 2010). Again, there is a need to provide students with a combination of curricular expectations, instructional practices, and pedagogical experiences that encourage them to construct and adapt critical thinking and reasoning skills when solving word problems in mathematics.

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