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Some Fixed Point Theorems of A_G-Contraction Mappings in G-metric Space

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Abstract: In this paper, we introduce a contraction mapping in G-metric space called A_G-contraction, which contains some of the contraction mappings that were previously defined by various authors. Further, we proved some fixed point theorems using A_G-contraction in complete G-metric space. Furthermore, we proved that our results generalize and improve some results of Mustafa and others.

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1 Introduction

The study of Banach contraction principle have led to number of generalizations and modification of the principle. It concerns certain mappings of a complete metric space into itself. It states sufficient condition for the existences and uniqueness of fixed points. The theorem also gives an iterative process by which we can obtain the approximation to the fixed points. Many authers have generalized the well known Banach contraction principle in several different forms, we may see for example [6, 8, 9, 12, 14, 16].

In [5] Dhage introduced D-metric space as a generalization of metric space and proved many results in this setting. But in 2006, Mustafa and Sims [11] proved that these results are not true in topological structure and hence they introduced G-metric space as a generalized form of metric space. Since then many authors have been studying fixed point results in G-metric spaces. In [1] Akram, Siddiqui and Zafar introduced a class of contractions, called *A*-contractions and proved some fixed theorems for self maps using *A*-contractions. This general class of contractions properly contains some of the contractions studied by Bianchini [4], Kannan [7], Khan [8] and Reich [13] for details see [1]. Further, Akram, Siddiqui and Zafar have studied some fixed point theorems using *A*-contraction in generalized metric spaces (gms), for detail see [2] and [3]. In this paper, we introduce a contraction mapping called A_G-contraction in G-metric space and prove some fixed point theorems for self mapping using A_G-contraction in G-metric space. Our results generalize and improve some results of Mustafa, Shatanawi and Bataineh [10] and others.

2 Preliminaries

In this section, we give some basic definitions and results on G-metric space from [11], which we require in the sequel.

Definition 2.1 *Let* X *be a nonempty set and let* $G : X \times X \times X \to \mathbb{R}_+$ *be a function satisfying the following properties,*

1. $G(x, y, z) = 0$ if $x = y = z$,
2. $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots\ldots$(symmetry in all three variables),
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangular inequality).

Then the function G is called a generalized metric or more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 2.2 *A* G *-metric space* (X, G) *is called symmetric* G *-metric if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$.

Definition 2.3 *Let* (X,G) *be a* G *-metric space, and* (x_n) *be a sequence of points of* X *, a point* $x \in X$ *is said to be the limit of the sequence* (x_n) *if* $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$*, and one can says that the sequence* (x_n) *is* G *-convergent to* x *.*

Thus, if $x_n \to x$ in a G-metric space (X,G), then for any $\varepsilon > 0$, there exist $N \in \mathrm{N}$ such that $\mathrm{G}(x, x_n, x_m) < \varepsilon$, for all $n,m \geq N$.

Proposition 2.4 *Let* (X,G) *be a G-metric space, then the following are equivalent,*

1. (x_n) is G-convergent to x,
2. $G(x_n, x_n, x) \to 0$, as $n \to \infty$,
3. $G(x_n, x, x) \to 0$, as $n \to \infty$,
4. $G(x_m, x_n, x) \to 0$, as $m,n \to \infty$.

Definition 2.5 *Let* (X,G) *be a* G *-metric space, a sequence* (x_n) *is called* G *-Cauchy if for every* $\varepsilon > 0$*, there is* $N \in \mathrm{N}$ *such that* $G(x_n, x_m, x_l) < \varepsilon$*, for* $n,m,l \geq N$ *; that is, if* $G(x_n, x_m, x_l) \to 0$ *as* $n,m,l \to \infty$ *.*

Proposition 2.6 *Let* (X,G) *be a* G *-metric space, then the following are equivalent,*

1. (x_n) is G-Cauchy,
2. for $\varepsilon > 0$, there exist $N \in \mathrm{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n,m \geq N$.

Definition 2.7 *A* G *-metric space* (X,G) *is said to be* G *-complete if every* G *-Cauchy sequence in* (X,G) *is* G *-convergent in* X *.*

Definition 2.8 *Let* (X,G) *and* (X', G') *be* G *-metric spaces and let* $f : (X,G) \to (X', G')$ *be a function, then* f *is said to be* G *-continuous at a point* $a \in X$ *if and only if, given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $x, y \in X$ *; and* $G(a, x, y) < \delta$ *implies* $G'(f(a), f(x), f(y)) < \varepsilon$*. A function* f *is* G *-continuous at* X *if and only if it is* G *-continuous at all* $a \in X$ *.*

3 A_G-Contraction and Its Comparison with Some Other Contractions

Definition 3.1 $\mathbb{R}_+$ *denotes the set of non-negative real numbers. Let* A *stands for the set of all functions* $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ *satisfying,*

1. α is G-continuous on the set $\mathbb{R}_+^3$ of all triplets of nonnegative reals (with respect to the Euclidean G-metric on $\mathbb{R}_+^3$).
2. $\alpha(0,0,0) = 0$.
3. $a \leq kb$, for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$, for all $a,b \in \mathbb{R}_+$.

Definition 3.2 ***(*** A_G ***-Contraction)***

A self map T on a G-metric X is said to be A_G-contraction of X if there exists $\alpha \in A$ such that,

$$G(Tx,Ty,Tz) \leq \alpha(G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)),$$

for all x,y,z in X.

Next, we will show that the following contractions are A_G-contraction.

1. There exist numbers $a,b,c \in [0,1)$ such that $a+b+c < 1$ and for all x,y,z in X,
$$G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz).$$
2. There exist a number $h \in [0,\frac{1}{3})$ such that for all x,y,z in X,
$$G(Tx,Ty,Tz) \leq h\{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)\}.$$

Theorem 3.3 *Let* $T : X \to X$ *be contraction defined as,*

$$G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz),$$

for all x,y,z in X and $a,b,c \in [0,1)$ such that $0 \leq a+b+c < 1$. Then T is an A_G-contraction.

Proof: Define $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ by $\alpha(u,v,w) = au + bv + cw$, for $a,b,c \in [0,1)$ and $0 \le a+b+c < 1$. Obviously α is continuous and $\alpha(0,0,0) = 0$. Also it is not difficult to show that $\alpha \in A$. Hence T is A_G-contraction.

Theorem 3.4 *Let $T : X \to X$ be contraction defined as,*

$$G(Tx,Ty,Tz) \le h\{G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz)\},$$

for all x,y,z in X and $h \in [0,\frac{1}{3})$. Then T is an A_G-contraction.

Proof: Define $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ by $\alpha(u,v,w) = h(u+v+w)$, for some $h \in [0,\frac{1}{3})$. Since addition and scalar multiplication are continuous so α is continuous and $\alpha(0,0,0) = 0$. Also it is easy to show that $\alpha \in A$. Hence T is A_G-contraction.

4 Some Fixed Point Theorems

Theorem 4.1 *Let (X,G) be a symmetric G-metric space and $T : X \to X$ be a mapping such that,*

1. $G(Tx,Ty,Tz) \le \alpha(G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz))$, for all $x,y,z \in X$,
2. T is G-continuous at a point $u \in X$,
3. there is $x \in X$ such that; $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ which G-converges to u.

Then u is a unique fixed point of T.

Proof: Given that T is G-continuous at a point $u \in X$, so $\{T^{n_i+1}(x)\}$ is G-convergent to Tu. We have to prove that $Tu = u$.

On contrary suppose that $Tu \ne u$. let $B_1 = B_G(u,\varepsilon)$ and $B_2 = B_G(Tu,\varepsilon)$ are the two G-open balls, where $\varepsilon = \frac{1}{6}G(u,Tu,Tu)$.

As $T^{n_i}(x) \to u$ and $T^{n_i+1}(x) \to Tu$, this implies that there is an $N_1 \in \mathbb{N}$ such that if $i > N_1$ implies $T^{n_i}(x) \in B_1$ and $T^{n_i+1}(x) \in B_2$. Thus our supposition implies that

$$G\left(T^{n_i}(x),T^{n_i+1}(x),T^{n_i+1}(x)\right) > \varepsilon \ldots (1),$$

for all $i > N_1$.

Using (i) and the fact that X is symmetric G-metric space, we can write

$$\begin{aligned}
G(T^{n_i}(x),T^{n_i+1}(x),T^{n_i+1}(x)) &= G(T(T^{n_i-1}(x)),T(T^{n_i}(x)),T(T^{n_i}(x))) \\
&= G(T(T^{n_i-1}(x)),T(T^{n_i-1}(x)),T(T^{n_i}(x))) \\
&\quad \le \alpha(G(T^{n_i-1}(x),T^{n_i}(x),T^{n_i}(x)),G(T^{n_i-1}(x),T^{n_i}(x),T^{n_i}(x)) \\
&\quad ,G(T^{n_i}(x),T^{n_i+1}(x),T^{n_i+1}(x))) \\
&\le kG(T^{n_i-1}(x),T^{n_i}(x),T^{n_i}(x))
\end{aligned}$$

Continuing in the same way, we get

$$G(T^{n_i}(x),T^{n_i+1}(x),T^{n_i+1}(x)) \le k^2 G(T^{n_i-2}(x),T^{n_i-1}(x),T^{n_i-1}(x)).$$

Now for $i > j > N_1 \in \mathrm{N}$, we have

$$\begin{aligned}
G(T^{n_i}(x),T^{n_i+1}(x),T^{n_i+1}(x)) &\le kG(T^{n_i-1}(x),T^{n_i}(x),T^{n_i}(x)) \\
&\le k^2 G(T^{n_i-2}(x),T^{n_i-1}(x),T^{n_i-1}(x)) \\
&. \\
&. \\
&. \\
&\le k^{n_i-n_j} G(T^{n_j}(x),T^{n_j+1}(x),T^{n_j+1}(x)).
\end{aligned}$$

So, as $i \to \infty$, we have $\lim G(T^{n_i}(x),T^{n_i+1}(x),T^{n_i+1}(x)) \le 0$. Which contradict (1). Hence $Tu = u$.

Now if $w \in X$ is another fixed point of T in X, that is, $Tw = w$, then we have

$$\begin{aligned}
G(w,u,u) &= G(Tw,Tu,Tu) \\
&\le \alpha(G(w,Tw,Tw),G(u,Tu,Tu),G(u,Tu,Tu)) \\
&\le \alpha(G(w,w,w),G(u,u,u),G(u,u,u)) \\
&\le \alpha(0,0,0) \\
&= 0.
\end{aligned}$$

This implies $w = u$. Hence the result is proved.

Following corollary is the Theorem 2.1 of [10].

Corollary 4.2 *Let* (X,G) *be a symmetric* G *-metric space and* $T: X \to X$ *be a mapping such that,*

1. $G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz)$ for all $x,y,z \in X$ and $0 < a+b+c < 1$,
2. T is G-continuous at a point $u \in X$,
3. there is $x \in X$ such that; $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ which G -converges to u. Then u is a unique fixed point of T.

Proof: Since we have proved that above contraction in (i) is A_G-contraction also we have proved in Theorem 4.1 that A_G-contraction along with condition (ii) and (iii) has a unique fixed point, hence it follows that the above contraction has also a unique fixed point.

Theorem 4.3 *Let* (X,G) *be a complete symmetric* G *-metric space and* $T: X \to X$ *be a* A_G *-contraction. Then* T *has a unique fixed point in* X *and for each* $x_0 \in X$ *the sequence of iterates* $\{T^n x_0\}$ *converges to this fixed point.*

Proof: Select $x_0 \in X$ and and define the iterative sequence $\{x_n\}$ by

$x_{n+1} = Tx_n$ (equivalently $x_n = T^n x_0$).

Now, $G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$.

Since X is symmetric G-metric space, we can write

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_n, Tx_{n-1}, Tx_{n-1}) \\ &= G(Tx_{n-1}, Tx_{n-1}, Tx_n) \end{aligned}$$

Using definition of A_G-contraction in G-metric, we can write

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq \alpha(G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n)) \\ &= \alpha(G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})) \\ &\leq kG(x_{n-1}, x_n, x_n) \\ &\leq k(kG(x_{n-2}, x_{n-1}, x_{n-1})) \\ &= k^2 G(x_{n-2}, x_{n-1}, x_{n-1}). \end{aligned}$$

Proceeding in the same way, we get $G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1)$. As $k < 1$, so when

$n \to \infty$, then $G(x_n, x_{n+1}, x_{n+1}) = 0$

Now, by repeated use of the rectangular inequality of G-metric, for every integer $P > 0$, we can write

$$\begin{aligned} G(x_n, x_{n+p}, x_{n+p}) \quad &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + G(x_{n+p-1}, x_{n+p}, x_{n+p}). \end{aligned}$$

This gives $\lim_{n\to\infty} G(x_n, x_{n+p}, x_{n+p}) = 0$, which implies $\{x_n\}$ is a G-Cauchy sequence and since X is complete there exist $x^{'} \in X$ such that $x_n \to x^{'}$ as $n \to \infty$.

Again, we have,

$$\begin{aligned} G(Tx_n, Tx_n, Tx^{'}) \quad &\leq \alpha(G(x_n, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n), G(x^{'}, Tx^{'}, Tx^{'})) \\ &\leq \alpha(G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x^{'}, Tx^{'}, Tx^{'})). \end{aligned}$$

Taking limit as $n \to \infty$ and by the continuity of α, we have

$$\begin{aligned} G(x^{'}, x^{'}, Tx^{'}) \quad &\leq \alpha(G(x^{'}, x^{'}, x^{'}), G(x^{'}, x^{'}, x^{'}), G(x^{'}, Tx^{'}, Tx^{'})) \\ &= \alpha(0,0, G(x^{'}, x^{'}, Tx^{'})) \\ &= k0 \\ &= 0. \end{aligned}$$

Thus $Tx^{'} = x^{'}$.

Now, if $w \in X$ is another fixed point of T in X, that is, $Tw = w$, then we have

$$\begin{aligned} G(w, x^{'}, x^{'}) \quad &= G(Tw, Tx^{'}, Tx^{'}) \\ &\leq \alpha(G(w, Tw, Tw), G(x^{'}, Tx^{'}, Tx^{'}), G(x^{'}, Tx^{'}, Tx^{'})) \\ &\leq \alpha(G(w, w, w), G(x^{'}, x^{'}, x^{'}), G(x^{'}, x^{'}, x^{'})) \\ &\leq \alpha(0,0,0) \\ &= 0. \end{aligned}$$

Thus $w = x^{'}$. This completes the proof.

In comparison of Theorem 4.3 with Theorem 4.1, we observe that in Theorem 4.1 we have omitted the completeness property of the G-metric space and instead we have assumed conditions (ii) and (iii). However, Example 2.4 of [10] support that conditions (ii) and (iii) in Theorem 4.1 do not guarantee the completeness of the G-metric space. The following corollary is the Theorem 2.3 of [10].

Corollary 4.4 *Let (X,G) be a complete G-metric space and $T:X\to X$ be a mapping satisfies for all x,y,z in X,*

$$G(Tx,Ty,Tz)\le aG(x,Tx,Tx)+bG(y,Ty,Ty)+cG(z,Tz,Tz),$$

where $o<a+b+c<1$, then T has a unique fixed point.

Proof: Since we have proved that above contraction is A_G-contraction, also we have proved in Theorem 4.3 that A_G-contraction has a unique fixed point in complete G-metric space, hence it follows that the above contraction has a unique fixed point.

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