

Unique Common Fixed Point of Sequences of Mappings in G-Metric Space

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Abstract. In this paper, we prove some fixed point theorems for sequences of self mappings using A -type contraction in G -metric space. we, also, show that these results extends and improves the corresponding results in [2, 14] and others.

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1. Introduction

The study of Banach contraction principle have led to number of generalizations and modification of the principle. It concerns certain mappings of a complete metric space into itself. It states sufficient condition for the existences and uniqueness of fixed points. The theorem also gives an iterative process by which we can obtain the approximation to the fixed points. Many authers have generalized the well known Banach contraction principle in several different forms, we may see for example [7, 9, 10, 13, 14,16, 18].

In [6] Dhage introduced D-metric space as a generalization of metric space and proved many results in this setting. But in 2005, Z. Mustafa and B. Sims [12] proved that these results are not true in topological structure and hence they introduced G -metric space as a generalized form of metric space. Since then many authors including Z. Mustafa [11] have been studying fixed point results in G -metric spaces. In [2] M. Akram, A. A. Siddiqui and A. A. Zafar introduced a class of contractions, called A -contractions and proved some fixed point theorems for self maps using A -contractions. This general class of contractions properly contains some of the contractions studied by R. Kannan [8], Bianchini [5], M. S. Khan [9] and Reich [15] for details see [2, 17]. Further, M Akram, A. A. Siddiqui and A. A. Zafar have studied some fixed point theorems using A -contraction in generalized metric spaces (gms), for detail see [3] and [4]. In this paper, we prove some fixed point theorems for sequences of self mappings using A -type contraction in G -metric space. Also, we show that these results extends and improves the corresponding results in [2, 14] and other corresponding results in the current literature.

2. Preliminaries

In this section, we give some basic definitions and results on G -metric space from [12], which we require in the sequel.

Definition 2.1 Let X be a nonempty set and let $G: X \times X \times X \rightarrow \mathbb{R}_+$ be a function satisfying the following properties,

1. $G(x, y, z) = 0$ if $x = y = z$,
2. $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangular

inequality).

Then the function G is called a generalized metric or more specifically, a G -metric on X , and the pair (X, G) is called a G -metric space.

Definition 2.2 A G -metric space (X, G) is called symmetric G -metric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.3 Let (X, G) be a G -metric space, and (x_n) be a sequence of points of X , a point $x \in X$ is said to be the limit of the sequence (x_n) if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, and one can say that the sequence (x_n) is G -convergent to x .

Thus, if $x_n \rightarrow x$ in a G -metric space (X, G) , then for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 2.4 Let (X, G) be a G -metric space, then the following are equivalent,

1. (x_n) is G -convergent to x ,
2. $G(x_n, x_n, x) \rightarrow 0$, as $n \rightarrow \infty$,
3. $G(x_n, x, x) \rightarrow 0$, as $n \rightarrow \infty$,
4. $G(x_m, x_n, x) \rightarrow 0$, as $m, n \rightarrow \infty$.

Definition 2.5 Let (X, G) be a G -metric space, a sequence (x_n) is called G -Cauchy if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for $n, m, l \geq N$; that is, if $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Proposition 2.6 Let (X, G) be a G -metric space, then the following are equivalent,

1. (x_n) is G -Cauchy,

2. for $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Definition 2.7 A G -metric space (X, G) is said to be G -complete if every G -Cauchy sequence in (X, G) is G -convergent in X .

Definition 2.8 Let (X, G) and (X', G') be G -metric spaces and let $f : (X, G) \rightarrow (X', G')$ be a function, then f is said to be G -continuous at a point $a \in X$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G -continuous at X if and only if it is G -continuous at all $a \in X$.

3. Fixed point theorems for sequences of self mapping

Definition 3.1 [2] Let A stands for the set of all functions $\alpha : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ satisfying,

1. α is G -continuous on the set \mathbb{R}_+^3 of all triplets of nonnegative reals (with respect to the Euclidean G -metric on \mathbb{R}_+^3).
2. $a \leq kb$, for some $k \in [0, 1)$, whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$, for all $a, b \in \mathbb{R}_+$.

Definition 3.2 (A-type Contraction) A self map T on a G -metric X is said to be A-type contraction of X if there exists $\alpha \in A$ such that,

$$G(Tx, Ty, Ty) \leq \alpha(G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty)),$$

for all x, y in X .

Next theorem is the extension of Theorem 6 of [2] from metric space setup to the G -metric space setup.

Theorem 3.3 Let $\alpha \in A$ and $\{T_n\}_{n=1}^\infty$ be a sequence of self mappings on a complete G -metric space (X, G) such that

$$G(T_i x, T_j y, T_j y) \leq \alpha(G(x, y, y), G(x, T_i x, T_i x), G(y, T_j y, T_j y)) \dots (1).$$

Then $\{T_n\}_{n=1}^\infty$ has a unique common fixed point in X .

Proof: Define a sequence $\{x_n\}$ in X as $x_n = T_n x_{n-1}$, where $n = 1, 2, 3, \dots$

Now,

$$G(x_1, x_2, x_2) = G(T_1 x_0, T_2 x_1, T_2 x_1).$$

By using (1), we get

$$\begin{aligned} G(x_1, x_2, x_2) &\leq \alpha(G(x_0, x_1, x_1), G(x_0, T_1 x_0, T_1 x_0), G(x_1, T_2 x_1, T_2 x_1)) \\ &= \alpha(G(x_0, x_1, x_1), G(x_0, x_1, x_1), G(x_1, x_2, x_2)) \\ &\leq kG(x_0, x_1, x_1), \text{ for some } k \in [0, 1) \dots (2). \end{aligned}$$

Again,

$$\begin{aligned}
G(x_2, x_3, x_3) &= G(T_2x_1, T_3x_2, T_3x_2) \\
&\leq \alpha(G(x_1, x_2, x_2), G(x_1, T_2x_1, T_2x_1), G(x_2, T_3x_2, T_3x_2)) \\
&\leq \alpha(G(x_1, x_2, x_2), G(x_1, x_2, x_2), G(x_2, x_3, x_3)) \\
&\leq kG(x_1, x_2, x_2), \text{ for some } k \in [0, 1).
\end{aligned}$$

Now using (2), we have

$$\begin{aligned}
G(x_2, x_3, x_3) &\leq k(kG(x_0, x_1, x_1)) \\
&\leq k^2G(x_0, x_1, x_1).
\end{aligned}$$

Similarly,

$$G(x_3, x_4, x_4) \leq k^3G(x_0, x_1, x_1).$$

Continuing in this way, we get, $G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1)$. As $k < 1$, $G(x_n, x_{n+1}, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$.

Now, by repeated use of the rectangular inequality of G -metric, for every integer $p > 0$, we can write

$$\begin{aligned}
G(x_n, x_{n+p}, x_{n+p}) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{n+p-1}, x_{n+p}, x_{n+p}).
\end{aligned}$$

This gives $\lim_{n \rightarrow \infty} G(x_n, x_{n+p}, x_{n+p}) = 0$, which implies $\{x_n\}$ is a G -Cauchy sequence and since X is complete there exist $x \in X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$.

Now for $n > m$, we can write

$$\begin{aligned}
G(x, T_n x, T_n x) &\leq G(x, x_{m+1}, x_{m+1}) + G(x_{m+1}, T_n x, T_n x) \\
&= G(x, x_{m+1}, x_{m+1}) + G(T_{m+1} x_m, T_n x, T_n x) \\
&\leq G(x, x_{m+1}, x_{m+1}) + \alpha(G(x_m, x, x), G(x_m, T_{m+1} x_m, T_{m+1} x_m), \\
&\quad G(x, T_n x, T_n x)) \\
&= G(x, x_{m+1}, x_{m+1}) + \alpha(G(x_m, x, x), G(x_m, x_{m+1}, x_{m+1}), \\
&\quad G(x, T_n x, T_n x)).
\end{aligned}$$

Let $m \rightarrow \infty$, then

$$\begin{aligned}
G(x, T_n x, T_n x) &\leq G(x, x, x) + \alpha(G(x, x, x), G(x, x, x), G(x, T_n x, T_n x)) \\
&\leq 0 + \alpha(0, 0, G(x, T_n x, T_n x)) \\
&\leq k0 \\
&= 0.
\end{aligned}$$

Which gives $x = T_n x$.

Now suppose y is another fixed point of T_n , that is, $T_n y = y$ for some $y \in X$.

$$G(x, y, y) = G(T_n x, T_n y, T_n y).$$

By using (1), we get

$$\begin{aligned}
G(x, y, y) &\leq \alpha(G(x, y, y), G(x, T_n x, T_n x), G(y, T_n y, T_n y)) \\
&= \alpha(G(x, y, y), G(x, x, x), G(y, y, y)) \\
&\leq \alpha(G(x, y, y), 0, 0) \\
&\leq k0 \\
&= 0.
\end{aligned}$$

Hence, $x = y$.

Corollary 3.4 Let $\alpha \in A$ and $\{T_n\}_{n=1}^{\infty}$ be a sequence of self mappings on a complete G -metric space (X, G) such that any one of the following contractive condition is satisfied,

1. There exist a number $a \in [0, \frac{1}{2})$ such that for all x, y in X ,

$$G(T_i x, T_j y, T_j y) \leq a(G(x, T_i x, T_i x) + G(y, T_j y, T_j y)).$$

2. There exist a number $h \in [0, 1)$ such that for all x, y in X ,

$$G(T_i x, T_j y, T_j y) \leq h \sqrt{G(x, T_i x, T_i x) G(y, T_j y, T_j y)}.$$

3. There exist a number $h \in [0, 1)$ such that for all x, y in X ,

$$G(T_i x, T_j y, T_j y) \leq h \max\{G(x, T_i x, T_i x), G(y, T_j y, T_j y)\}.$$

4. There exist numbers $a, b, c \in [0, 1)$ such that $a + b + c < 1$ and for all x, y in

X ,

$$G(T_i x, T_j y, T_j y) \leq aG(x, y, y) + bG(x, T_i x, T_i x) + cG(y, T_j y, T_j y).$$

Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X .

Proof: In [1], it is shown that above contractions are A -type contraction so by Theorem 3.3, we can conclude that $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in X .

Next theorem is analogous to the Theorem 2 of [3] in G -metric space.

Theorem 3.5 Let $\{T_m\}$ and $\{S_n\}$ be sequences of self maps on a complete symmetric G -metric space (X, G) satisfying,

$$G(T_m x, S_n y, S_n y) \leq \alpha(G(T_m x, x, x), G(S_n y, y, y), G(x, y, y)),$$

for all $x, y \in X$, for some $\alpha \in A$, and for each $m, n \in \mathbb{N}$. Then $\{T_m\}$ and $\{S_n\}$ have a unique common fixed point.

Proof: Define a sequence $\{x_n\}$ in X as, $x_{2n+1} = T_{n+1} x_{2n}$ and $x_{2n} = S_n x_{2n-1}$.

Consider,

$$\begin{aligned}
G(x_{2n}, x_{2n+1}, x_{2n+1}) &= G(S_n x_{2n-1}, T_{n+1} x_{2n}, T_{n+1} x_{2n}) \\
&\leq \alpha(G(S_n x_{2n-1}, x_{2n-1}, x_{2n-1}), G(T_{n+1} x_{2n}, x_{2n}, x_{2n}), \\
&\quad G(x_{2n-1}, x_{2n}, x_{2n})) \\
&= \alpha(G(x_{2n}, x_{2n-1}, x_{2n-1}), G(x_{2n+1}, x_{2n}, x_{2n}), G(x_{2n-1}, x_{2n}, x_{2n})).
\end{aligned}$$

Since X is symmetric G -metric Space, we can write

$$\begin{aligned} G(x_{2n}, x_{2n+1}, x_{2n+1}) &\leq \alpha(G(x_{2n-1}, x_{2n}, x_{2n}), G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n-1}, x_{2n}, x_{2n})) \\ &\leq kG(x_{2n-1}, x_{2n}, x_{2n}), \end{aligned}$$

for some $k \in [0,1)$. Similarly, we have

$$G(x_{2n-1}, x_{2n}, x_{2n}) \leq kG(x_{2n-2}, x_{2n-1}, x_{2n-1}), \text{ for some } k \in [0,1).$$

Which gives

$$G(x_{2n}, x_{2n+1}, x_{2n+1}) \leq k^2 G(x_{2n-2}, x_{2n-1}, x_{2n-1}).$$

Proceeding in the same way, we get

$$G(x_{2n+1}, x_{2n}, x_{2n}) \leq k^{2n} G(x_0, x_1, x_1).$$

In general, we have

$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1)$, for some $k \in [0,1)$. As $k < 1$, $G(x_n, x_{n+1}, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$.

Now, by repeated use of the rectangular inequality of G -metric, for every integer $p > 0$, we can write

$$\begin{aligned} G(x_n, x_{n+p}, x_{n+p}) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{n+p-1}, x_{n+p}, x_{n+p}). \end{aligned}$$

This gives $\lim_{n \rightarrow \infty} G(x_n, x_{n+p}, x_{n+p}) = 0$, which implies $\{x_n\}$ is a G -Cauchy sequence and since X is complete there exist $x \in X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$.

Now for each $m > n$ consider,

$$\begin{aligned} G(T_m x, x, x) &\leq G(T_m x, x_{2n}, x_{2n}) + G(x_{2n}, x, x) \\ &\leq G(x_{2n}, x, x) + G(T_m x, x_{2n}, x_{2n}) \\ &\leq G(x_{2n}, x, x) + G(T_m x, g_n x_{2n-1}, g_n x_{2n-1}) \\ &\leq G(x_{2n}, x, x) + \alpha(G(T_m x, x, x), G(g_n x_{2n-1}, x_{2n-1}, x_{2n-1}), \\ &\quad G(x, x_{2n-1}, x_{2n-1})) \\ &= G(x_{2n}, x, x) + \alpha(G(T_m x, x, x), G(x_{2n}, x_{2n-1}, x_{2n-1}), \\ &\quad G(x, x_{2n-1}, x_{2n-1})). \end{aligned}$$

Let $n \rightarrow \infty$, we get

$$\begin{aligned} G(T_m x, x, x) &\leq G(x, x, x) + \alpha(G(T_m x, x, x), G(x, x, x), G(x, x, x)) \\ &\leq 0 + \alpha(G(T_m x, x, x), 0, 0) \\ &\leq k0 \\ &= 0. \end{aligned}$$

This implies that $x = T_m x$. Similarly, we can show that $S_n x = x$. Thus $\{T_m\}$ and $\{S_n\}$ have common fixed point x . The uniqueness can be obtain easily. This completes the proof.

Corollary 3.6 Let $\alpha \in A$ and $\{T_n\}$ and $\{S_n\}$ be a sequences of self mappings on a complete G -metric space (X, G) such that any one of the following contractive condition is satisfied,

1. There exist a number $a \in [0, \frac{1}{2})$ such that for all x, y in X ,

$$G(T_m x, S_n y, S_n y) \leq a(G(x, T_m x, T_m x) + G(y, S_n y, S_n y)).$$
2. There exist a number $h \in [0, 1)$ such that for all x, y in X ,

$$G(T_m x, S_n y, S_n y) \leq h \sqrt{G(x, T_m x, T_m x)G(y, S_n y, S_n y)}.$$
3. There exist a number $h \in [0, 1)$ such that for all x, y in X ,

$$G(T_m x, S_n y, S_n y) \leq h \max\{G(x, T_m x, T_m x), G(y, S_n y, S_n y)\}.$$
4. There exist numbers $a, b, c \in [0, 1)$ such that $a + b + c < 1$ and for all x, y in

X ,

$$G(T_m x, S_n y, S_n y) \leq aG(x, y, y) + bG(x, T_m x, T_m x) + cG(y, S_n y, S_n y).$$

Then $\{T_n\}$ and $\{S_n\}$ has a unique common fixed point in X .

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