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**Calculation of Hyperbolic Flow Past a Circular Cylinder
using Indirect Boundary Element Method in Case of
Constant Variation**

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Abstract:

In present paper, indirect boundary element method (IDBEM) has been applied to calculate Hyperbolic flow past a circular cylinder in case of constant variation. The boundary of the circular cylinder (C.C) is discretized into constant boundary elements over which the velocity distribution is calculated. The calculated results are also compared with exact results.

Keywords:

Indirect boundary element method, Hyperbolic flow, Circular cylinder, Constant variation.

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1. Introduction

Boundary element method is a numerical technique used to solve the different types of problems today facing in science and technology. The well-known computational methods such as finite difference method (FDM) and finite element method (FEM) are very costly and time-consuming because in these methods the whole domain under study is discretised into a number of block-type elements, whereas in boundary element method the process of discretisation takes place on the surface of body. Which considerably reduces the size of system of equations resulting the reduction in data and that is requisite to run a computer program efficiently. In other words, boundary element methods are superior in several aspects to other computational methods because of their surface modeling approach. That is why; the complicated structures can be more easily modeled by these methods and are therefore preferred by engineers. The results of boundary element methods are more accurate and reliable than those of classical methods. Which establishes the fact that these methods (BEMs) are time-saving, accurate, efficient and economical techniques as compared to other numerical techniques (Mushtaq; 2008, 2009). These salient features of BEMs make them popular in communities of engineering and science. Such methods are essentially the methods for solving the partial differential equations arising in wide range of fields, e.g., fluid mechanics, solid and fracture mechanics, heat transfer and electromagnetic theory, potential theory, elasticity, elastostatics and elastodynamics, etc. as detailed in Brebbia and Walker, (1980). Furthermore, the area of their applications is increasing day to day. Boundary element methods have been classified into direct and indirect methods. The direct method is in the form of a statement which gives the values of unknown variables at the field point under discussion in terms of a complete set of the entire boundary data. Where as the indirect method is based on the distribution of sources or doublets over the boundary of the body and calculates such distribution in terms of the solution of an integral equation. The IDBEM has been used for many years in the past for flow field calculations due to its simplicity. The first work on flow field calculations around three-dimensional bodies was probably done by Hess and Smith, (1962 & 1967). The DBEM for potential flow calculations around objects was first applied in past by Morino et al, (1975). In recent past, the IDBEM's have been applied by the author himself for flow field calculations around two- and three-dimensional objects (Muhammad; 2008, 2010,2014).

2. Calculation of Hyperbolic Flow past a Circular Cylinder

Boundary element methods are applied for both problems of exterior and interior flows in two dimensional space. In this case, DBEM is used to calculate the Oseen's flow around a C.C using constant variation. A circular cylinder of radius 'a' is held fixed in a uniform stream of incompressible viscous fluid flowing steadily around it and let the centre of a cylinder be taken as origin and U_s be the velocity of uniform stream in the positive x – direction as given in figure 1.(Chwang & T.Y.Wu,1975; Shah,2008; Muhammad, 2014).

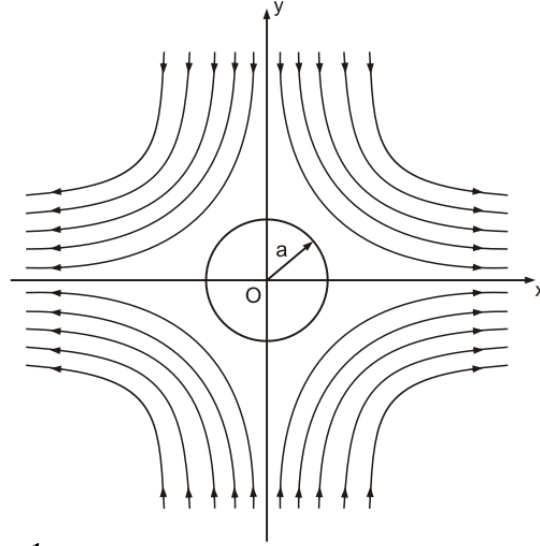


Figure 1: Hyperbolic flow past a circular cylinder

The stream function for such flow is given by

$$\begin{aligned}\psi &= \Omega a^2 x y \left(1 - \frac{a^2}{r^2}\right)^2 \\ &= U a^2 \frac{x y}{\sqrt{x^2 + y^2}} \left(1 - \frac{a^2}{r^2}\right)^2\end{aligned}\quad (1)$$

$$u = -\frac{\partial \psi}{\partial y}$$

$$\text{and } v = \frac{\partial \psi}{\partial x}$$

$$\begin{aligned}\frac{\partial \psi}{\partial y} &= U a^2 \left[\left\{ \frac{x}{\sqrt{x^2 + y^2}} - \frac{x y (2 y)}{2 (x^2 + y^2)^{3/2}} \right\} \left(1 - \frac{a^2}{r^2}\right)^2 + \frac{x y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y} \left(1 - \frac{a^2}{r^2}\right)^2 \right] \\ &= U a^2 \left[\left\{ \frac{x (x^2 + y^2) - x y^2}{(x^2 + y^2)^{3/2}} \right\} \left(1 - \frac{a^2}{r^2}\right)^2 + \frac{x y}{\sqrt{x^2 + y^2}} 2 \left(1 - \frac{a^2}{r^2}\right) \frac{\partial}{\partial y} \left(1 - \frac{a^2}{r^2}\right) \right] \\ &= U a^2 \left[\left\{ \frac{x^3 + x y^2 - x y^2}{(x^2 + y^2)^{3/2}} \right\} \left(1 - \frac{a^2}{r^2}\right)^2 + \frac{2 x y}{\sqrt{x^2 + y^2}} \left(1 - \frac{a^2}{r^2}\right) \left(\frac{2 a^2}{r^3} \frac{\partial r}{\partial y} \right) \right] \\ &= U a^2 \left[\frac{x^3}{(x^2 + y^2)^{3/2}} \left(1 - \frac{a^2}{r^2}\right)^2 + \frac{2 x y}{\sqrt{x^2 + y^2}} \left(1 - \frac{a^2}{r^2}\right) \frac{2 a^2}{r^3} \frac{y}{r} \right] \\ &= U a^2 \left[\frac{x^3}{(x^2 + y^2)^{3/2}} \left(1 - \frac{a^2}{r^2}\right)^2 + \frac{4 a^2 x y^2}{r^4 \sqrt{x^2 + y^2}} \left(1 - \frac{a^2}{r^2}\right) \right]\end{aligned}$$

$$v = -\frac{\partial \psi}{\partial x}$$

$$= -U a^2 \left[\frac{x^3}{(x^2 + y^2)^{3/2}} \left(1 - \frac{a^2}{r^2}\right)^2 + \frac{4 a^2 x y^2}{r^4 \sqrt{x^2 + y^2}} \left(1 - \frac{a^2}{r^2}\right) \right]$$

$$\begin{aligned}
 &= -U a^2 \left[\frac{x^3}{r^3} \left(\frac{r^2 - a^2}{r^2} \right)^2 + \frac{4 a^2 x y^2}{r^5} \left(\frac{r^2 - a^2}{r^2} \right) \right] \\
 &= -U a^2 \left[\frac{x^3 (r^2 - a^2)^2}{r^7} + \frac{4 a^2 x y^2 (r^2 - a^2)}{r^7} \right] \\
 &= -\frac{U a^2 (r^2 - a^2)}{r^7} [x^3 (r^2 - a^2) + 4 a^2 x y^2] \quad (2)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \Psi}{\partial x} &= U a^2 \left[\left\{ \frac{y}{\sqrt{x^2 + y^2}} - \frac{x y (2 x)}{2 (x^2 + y^2)^{3/2}} \right\} \left(1 - \frac{a^2}{r^2} \right)^2 + \frac{x y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} \left(1 - \frac{a^2}{r^2} \right)^2 \right] \\
 &= U a^2 \left[\left\{ \frac{y(x^2 + y^2) - x^2 y}{(x^2 + y^2)^{3/2}} \right\} \left(1 - \frac{a^2}{r^2} \right)^2 + \frac{x y}{\sqrt{x^2 + y^2}} 2 \left(1 - \frac{a^2}{r^2} \right) \frac{\partial}{\partial x} \left(1 - \frac{a^2}{r^2} \right) \right] \\
 &= U a^2 \left[\left\{ \frac{x^2 y - y^3 - x^2 y}{(x^2 + y^2)^{3/2}} \right\} \left(1 - \frac{a^2}{r^2} \right)^2 + \frac{2 x y}{\sqrt{x^2 + y^2}} \left(1 - \frac{a^2}{r^2} \right) \left(\frac{2 a^2 x}{r^3} \frac{1}{r} \right) \right] \\
 &= U a^2 \left[\frac{y^3}{(x^2 + y^2)^{3/2}} \left(1 - \frac{a^2}{r^2} \right)^2 + \frac{4 a^2 x^2 y}{r^4 \sqrt{x^2 + y^2}} \left(1 - \frac{a^2}{r^2} \right) \right] \\
 &= U a^2 \left[\frac{y^3}{r^3} \left(\frac{r^2 - a^2}{r^2} \right)^2 + \frac{4 a^2 x^2 y}{r^5} \left(\frac{r^2 - a^2}{r^2} \right) \right] \\
 &= U a^2 \left[\frac{y^3 (r^2 - a^2)^2}{r^7} + \frac{4 a^2 x^2 y (r^2 - a^2)}{r^7} \right] \\
 v &= \frac{U a^2 (r^2 - a^2)}{r^7} [y^3 (r^2 - a^2) + 4 a^2 x^2 y] \quad (3)
 \end{aligned}$$

The magnitude of velocity is given by the relation

$$V = \sqrt{u^2 + v^2} \quad (4)$$

The equation for the indirect method in the case of a doublet distribution for the problems of two-dimensional exterior flow is given by

$$-\frac{1}{2} \Phi_i + \frac{1}{2\pi} \int_{\Gamma-i} \Phi \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) d\Gamma + \phi_\infty = -(\phi_{u.s})_i \quad (5)$$

Since $(\phi_{u.s})_i = -x_i$, the above equation becomes

$$-\frac{1}{2} \Phi_i + \frac{1}{2\pi} \int_{\Gamma-i} \Phi \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) d\Gamma + \phi_\infty = x_i \quad (6)$$

Now to approximate the surface of a C.C, the coordinates of extreme points on the BE's are generated in a computer program as under (Muhammad, G, 2008).

The surface of the C.C is discretized into m elements in a clockwise direction using the following formula

$$\theta_k = [(m+3) - 2k] \pi / m, \quad k = 1, 2, \dots, m \quad (7)$$

Then the coordinates of the extreme points of these m elements are calculated from

$$\left. \begin{aligned} x_k &= a \cos \theta_k \\ y_k &= a \sin \theta_k \end{aligned} \right\} \quad k = 1, 2, \dots, m \quad (8)$$

Take $m = 8$ and $a = 1$.

2.1 Constant Variation

Let us consider the constant element case in which nodal points are to be taken at the middle of each element. Also ϕ and $\frac{\partial \phi}{\partial n}$ are in this case constant over such elements and equal to the value at the mid-node of the element.

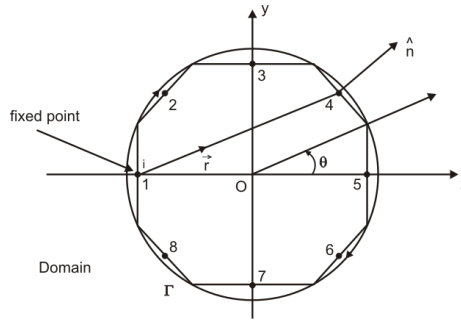


Figure 2

The discretization of the surface of a C.C is shown in figure2.

The mid-node coordinates over every element are defined by the formula .

$$\left. \begin{aligned} x_m &= \frac{x_k + x_{k+1}}{2} \\ y_m &= \frac{y_k + y_{k+1}}{2} \end{aligned} \right\} \quad k, m = 1, 2, \dots, 8 \quad (9)$$

2.2 Matrix Form

Equation (6) can be written as

$$-\frac{1}{2} \Phi_i + \sum_{j=1}^m \hat{H}_{ij} \Phi_j + \phi_\infty = x_i \quad (10)$$

$$\text{or } \sum_{j=1}^m H_{ij} \Phi_j + \phi_\infty = x_i \quad (11)$$

When all the nodal points are taken into consideration, which can be put in the matrix form i.e.

$$[H] \{U\} = \{R\}$$

Where $[H]$ is a matrix of influence coefficients, $\{U\}$ is a vector of unknown total potentials and $\{R\}$ on the R.H.S. is a known vector whose elements are the negative

values of the velocity potential of the uniform stream at the nodal points on the boundary of the C.C.

Since $\frac{\partial \phi}{\partial n}$ is specified at each node of the element, the values of the perturbation velocity potential ϕ can be found at each node on the boundary. The total potential Φ is then found, which will then be used to calculate the velocity on the C.C.

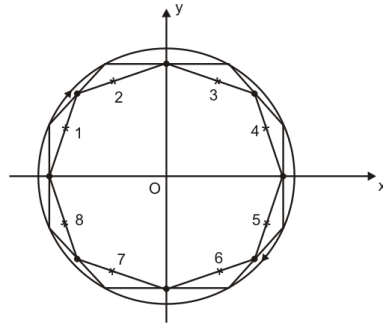


Figure 3

The velocity midway between two nodes on the boundary can then be approximated by using the formula

$$\text{Velocity } \bar{V} = \frac{\Phi_{k+1} - \Phi_k}{\text{Length from node } k \text{ to } k+1} \quad (12)$$

The method has been implemented using FORTRAN programming with 16, 32, and 64 constant BE's

TABLE (1)

ELEMENT	X	Y	VELOCITY	EXACT VELOCITY
1	-.94	.19	.39785E+00	.39197E+01
2	-.80	.53	.11330E+01	.76729E+01
3	-.53	.80	.16956E+01	.76729E+01
4	-.19	.94	.20001E+01	.39197E+01
5	.19	.94	.20001E+01	.39197E+01
6	.53	.80	.16956E+01	.76729E+01
7	.80	.53	.11330E+01	.76729E+01
8	.94	.19	.39785E+00	.39197E+01
9	.94	-.19	.39785E+00	.39197E+01
10	.80	-.53	.11330E+01	.76729E+01
11	.53	-.80	.16956E+01	.76729E+01
12	.19	-.94	.20001E+01	.39197E+01
13	-.19	-.94	.20001E+01	.39197E+01
14	-.53	-.80	.16956E+01	.76729E+01
15	-.80	-.53	.11330E+01	.76729E+01
16	-.94	-.19	.39785E+00	.39197E+01

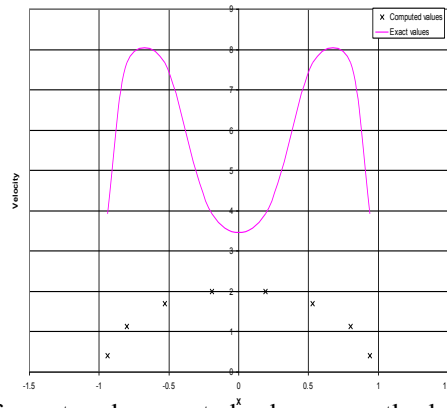


Figure 4: Comparison of exact and computed values over the boundary of a circular cylinder for 16 constant BE's.

TABLE (2)

ELEMENT	X	Y	VELOCITY	EXACT VELOCITY
1	-.99	.10	.19699E+00	.23991E+01
2	-.95	.29	.58339E+00	.42331E+01
3	-.87	.47	.94738E+00	.59262E+01
4	-.77	.63	.12750E+01	.68762E+01
5	-.63	.77	.15535E+01	.68762E+01
6	-.47	.87	.17724E+01	.59262E+01
7	-.29	.95	.19232E+01	.42330E+01
8	-.10	.99	.20001E+01	.23991E+01
9	.10	.99	.20001E+01	.23991E+01
10	.29	.95	.19232E+01	.42330E+01
11	.47	.87	.17724E+01	.59262E+01
12	.63	.77	.15535E+01	.68762E+01
13	.77	.63	.12750E+01	.68762E+01
14	.87	.47	.94738E+00	.59262E+01
15	.95	.29	.58339E+00	.42330E+01
16	.99	.10	.196998E+00	.23991E+01
17	.99	-.10	.19699E+00	.23991E+01
18	.95	-.29	.58339E+00	.42331E+01
19	.87	-.47	.94738E+00	.59262E+01
20	.77	-.63	.12750E+01	.68762E+01
21	.63	-.77	.15535E+01	.68762E+01
22	.47	-.87	.17724E+01	.59262E+01
23	.29	-.95	.19232E+01	.42330E+01
24	.10	-.99	.20001E+01	.23991E+01
25	-.10	-.99	.20001E+01	.23991E+01
26	-.29	-.95	.19232E+01	.42330E+01
27	-.47	-.87	.17724E+01	.59262E+01
28	-.63	-.77	.15535E+01	.68762E+01
29	-.77	-.63	.12750E+01	.68762E+01
30	-.87	-.47	.94738E+00	.59262E+01
31	-.95	-.29	.58339E+00	.42330E+01
32	-.99	-.10	.19699E+00	.23991E+01

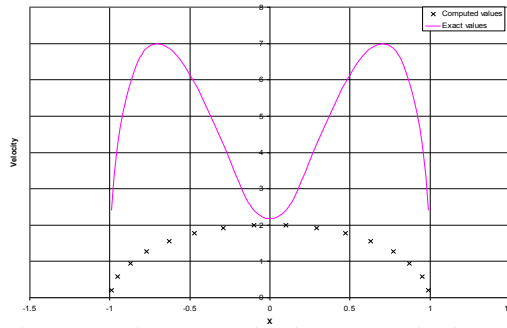


Figure 5: Comparison of exact and computed values over the boundary of a circular cylinder for 32 constant BE's.

TABLE (3)

ELEMENT	X	Y	VELOCITY	EXACT VELOCITY
1	-1.00	.05	.98258E-01	.20000E+01
2	-.99	.15	.29383E+00	.26642E+01
3	-.97	.24	.48654E+00	.35808E+01
4	-.94	.34	.67460E+00	.45056E+01
5	-.90	.43	.85616E+00	.53288E+01
6	-.86	.51	.10295E+01	.59895E+01
7	-.80	.59	.11929E+01	.64497E+01
8	-.74	.67	.13447E+01	.66855E+01
9	-.67	.74	.14837E+01	.66855E+01
10	-.59	.80	.16084E+01	.64497E+01
11	-.51	.86	.17175E+01	.59895E+01
12	-.43	.90	.18102E+01	.53288E+01
13	-.34	.94	.18854E+01	.45056E+01
14	-.24	.97	.19424E+01	.35808E+01
15	-.15	.99	.19808E+01	.26642E+01
16	-.05	1.00	.20000E+01	.20000E+01
17	.05	1.00	.20000E+01	.20000E+01
18	.15	.99	.19808E+01	.26642E+01
19	.24	.97	.19424E+01	.35808E+01
20	.34	.94	.18854E+01	.45056E+01
21	.43	.90	.18102E+01	.53288E+01
22	.51	.86	.17175E+01	.59895E+01
23	.59	.80	.16084E+01	.64497E+01
24	.67	.74	.14837E+01	.66855E+01
25	.74	.67	.13448E+01	.66855E+01
26	.80	.59	.11929E+01	.64497E+01
27	.86	.51	.10295E+01	.59895E+01
28	.90	.43	.85616E+00	.53288E+01
29	.94	.34	.67461E+00	.45056E+01
30	.97	.24	.48654E+00	.35808E+01
31	.99	.15	.29383E+00	.26642E+01
32	1.00	.05	.98258E-01	.20000E+01
33	1.00	-.05	.98258E-01	.20000E+01
34	.99	-.15	.29383E+00	.26642E+01
35	.97	-.24	.48654E+00	.35808E+01
36	.94	-.34	.67460E+00	.45056E+01
37	.90	-.43	.85616E+00	.53288E+01

38	.86	-.51	.10295E+01	.59895E+01
39	.80	-.59	.11929E+01	.64497E+01
40	.74	-.67	.13447E+01	.66855E+01
41	.67	-.74	.14837E+01	.66855E+01
42	.59	-.80	.16084E+01	.64497E+01
43	.51	-.86	.17175E+01	.59895E+01
44	.43	-.90	.18102E+01	.53288E+01
45	.34	-.94	.18854E+01	.45056E+01
46	.24	-.97	.19424E+01	.35808E+01
47	.15	-.99	.19808E+01	.26642E+01
48	.05	-1.00	.20000E+01	.20000E+01
49	-.05	-1.00	.20000E+01	.20000E+01
50	-.15	-.99	.19808E+01	.26642E+01
51	-.24	-.97	.19424E+01	.35808E+01
52	-.34	-.94	.18854E+01	.45056E+01
53	-.43	-.90	.18102E+01	.53288E+01
54	-.51	-.86	.17175E+01	.59895E+01
55	-.59	-.80	.16084E+01	.64497E+01
56	-.67	-.74	.14837E+01	.66855E+01
57	-.74	-.67	.13448E+01	.66855E+01
58	-.80	-.59	.11929E+01	.64497E+01
59	-.86	-.51	.10295E+01	.59895E+01
60	-.90	-.43	.85616E+00	.53288E+01
61	-.94	-.34	.67461E+00	.45056E+01
62	-.97	-.24	.48654E+00	.35808E+01
63	-.99	-.15	.29383E+00	.26642E+01
64	-1.00	-.05	.98258E-01	.20000E+01

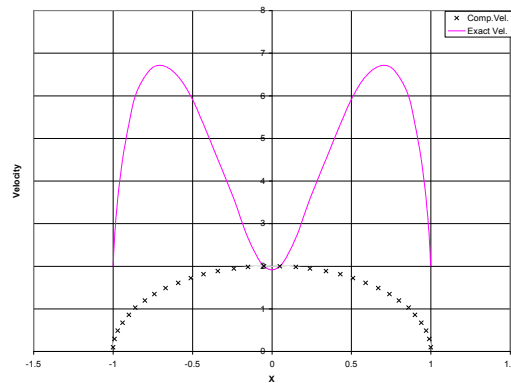


Figure 6: Comparison exact and computed values over the boundary of a circular cylinder for 64 constant BE's.

3. Conclusion

Indirect boundary element method has been applied to calculate the hyperbolic flow past a circular cylinder in case of constant variation. The improvement in results gained by taking 32 and 64 constant elements can be seen from the tables (2), (3) and figures 5, 6 and such improvement increases with increase in number of boundary elements. Moreover, the computed results are in good agreement with exact results at the top of a body under consideration.

References

- [1] Brebbia, C.A. and Walker, S.: "Boundary Element Techniques in Engineering", Newnes-Butterworths (1980).
- [2] Lamb, H.: "Hydrodynamics" 6th Edition, Cambridge University Press (1932).
- [3] Milne-Thomson, L.M.: "Theoretical Hydrodynamics", 5th Edition, London Macmillan & Co. Ltd. (1968).
- [4] Hess, J.L. & Smith, A.M.O.: "Calculation of non-lifting potential flow about arbitrary three-dimensional bodies", McDonnell Douglas Report No.E.S. 40622, (1962).
- [5] Hess, J.L. & Smith, A.M.O.: "Calculation of potential flow about arbitrary bodies", Progress in Aeronautical Sciences, Vol. 8, pp 1-158, Pergamon Press, (1967).
- [6] Muhammad, G. & N.A. Shah: "Calculation of Creeping Flow Past a Sphere Using Direct Boundary Element Method", Journal of American Science, Vol. 6, Issue 6, (2010).
- [7] Muhammad, G. & N.A. Shah: "Calculation of Creeping Flow Past a Sphere Using Indirect Boundary Element Method", Journal of American Science, Accepted, (2010).
- [8] Morino L., Chen, Lee-Tzong & Suci, E.O.: "A steady and oscillatory subsonic and supersonic aerodynamics around complex configuration", AIAA Journal, 13, 1975, 368-374.
- [9] Shah, N.A.: "Ideal Fluid Dynamics", A-One Publishers, Lahore-Pakistan (2008).
- [10] A.T. Chwang & T.Y. Wu.: "Hydromechanics of low-Reynolds-number flow", J. Fluid Mech. (1975), vol. 67, part 4, pp. 787-815.
- [11] Muhammad, G., Shah, N.A. & Mushtaq, M.: "Indirect Boundary Element Method for the Flow Past a Circular Cylinder with Linear element Approach", International Journal of Applied Engineering Research, India, Vol. 3, No. 12, Dec. (2008).
- [12] Mushtaq M, Shah, N.A. & Muhammad, G.: Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Circular Cylinder with Linear Element Approach", Australian Journal of Basic and Applied Sciences", Vol. 2, No. 4 (2008).
- [13] Mushtaq M, Shah, N.A. & Muhammad, G.: Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Circular Cylinder with Constant Element Approach", Journal of American Science", Vol. 5, No. 4 (2009).
- [14] Muhammad, G., Shah, N.A.; Mushtaq, M.; Nazir, S; & Anjam, Y.N.: "Indirect Boundary Element Method for Calculation of Hyperbolic Flow Past a Circular Cylinder in case of Linear Variation", Journal of Pakistan Institute of Chemical Engineers, Vol. 42, No. 1, (2014).
- [15] Muhammad, G.: "Boundary Element Methods for Incompressible Fluid Flow Problems", Ph.D. Thesis (2011), Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.