Assesment of Smarandache Curves in The Null Cone Q²

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Abstract. In this study, Smarandache curves according to the asymptotic orthonormal frame are given in null cone $\mathbf{Q}^2$. By using cone frame formulas, some characterizations of Smarandache curves are obtained and cone frenet invariants of these curves are calculated. Also, these curves are illustrated with an example.

1. INTRODUCTION

Curves arise naturally in numerous areas of the physical sciences and within areas of pure mathematics itself. The greatest effect in the research of curves was the discovery of the calculus. Broading speaking, the study of parametrized curves represents the beginning of a major area of mathematics called differential geometry. In differential gometry, there are many significant results and characteristics in the theory of the curves. Invertigaters pursue exertions regarding the curves. In the light of the available studies, authors always present new curves. In [2, 3], the author introduced some special Smarandache curves in the Euclidean space and the author gave Frenet-Serret invariants of a special case and the author defined a special case of such curves and call it Smarandache $TB2$ curves in the Minkowski space-time, respectively. Smarandache curves are one of them. Smarandache geometry is a geometry which has at any rate one Smarandachely disowned axiom in [4]. An axiom is said to be Smarandachely disowned, if it acts in at any rate two dissimilar ways insided of the same space. Thus, it is said that an axiom is partially negated, or there

is a degree of negation of an axiom. The most important contribution of Smarandache ge-
ometries was the introduction of the degree of an axiom which works somehow like the
negation in fuzzy logic or more general like the negation in neutrosophic logic. Smaran-
dache curves are the ones whose position vector is constituted by Frenet frame vectors of
the other regular curve. Smarandache curves in various ambient spaces have been classfied
in $[1], [5 - 14], [15, 16, 19], [21 - 27], [29 - 33]$ and $[28]$. In this paper, we define spe-
cial Smarandache curves such as $x\alpha, xy, \alpha y$ and $x\alpha y-$Smarandache curves according to
asymptotic orthonormal frame in the null cone $\mathbf{Q}^2$ and we examine the curvature and the
asymptotic orthonormal frame's vectors of Smarandache curves. We also give an example
related to these curves.

2. Notations and Preliminaries

Some basics of the curves in the null cone are provided from, $[20]$. Let E_1^3 be the
$3-$dimensional pseudo-Euclidean space with the

$$\widetilde{g}(X, Y) = \langle X, Y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$$

for all $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in E_1^3$. E_1^3 is a flat pseudo-Riemannian manifold
of signature $(2, 1)$. Let M be a submanifold of E_1^3. If the pseudo-Riemannian metric $\widetilde{g}$
of E_1^3 induces a pseudo-Riemannian metric g(respectively, a Riemannian metric, a degen-
erate quadratic form) on M, then M is called a timelike(respectively, spacelike, degen-
erate) submanifold of E_1^3. Let c be a fixed point in E_1^3. The pseudo-Riemannian lightlike
cone(quadric cone) is defined by

$$\mathbf{Q}_1^2(c) = \left\{ x \in E_1^3 : g(x - c, x - c) = 0 \right\},$$

where the point c is called the center of $\mathbf{Q}_1^2(c)$. When $c = 0$, we merely indicate $\mathbf{Q}_1^2(0)$ by
$\mathbf{Q}^2$ be and call it the null cone.

Let E_1^3 be $3-$dimensional Minkowski space and $\mathbf{Q}^2$ the lightlike cone in E_1^3. A vector
$V \neq 0$ in E_1^3 is called spacelike, timelike or lightlike, if $\langle V, V \rangle > 0$, $\langle V, V \rangle < 0$ or
$\langle V, V \rangle = 0$, respectively. The norm of a vector $x \in E_1^3$ is given by $\|x\| = \sqrt{\langle x, x \rangle}$, $[20]$.

We suppose that curve $x : I \to \mathbf{Q}^2 \subset E_1^3$ is a regular curve in $\mathbf{Q}^2$ for $t \in I$. Below, we
always suppose that the curve is regular.

A frame field $\{x, \alpha, y\}$ on E_1^3 is called an asymptotic orthonormal frame field, if

$$\langle x, x \rangle = \langle y, y \rangle = \langle x, \alpha \rangle = \langle y, \alpha \rangle = 0, \ \langle x, y \rangle = \langle \alpha, \alpha \rangle = 1.$$

Using $x'(s) = \alpha(s)$ we have that $\{x(s), \alpha(s), y(s)\}$ from an asymptotic orthonormal
frame throughout the curve $x(s)$ and the cone frenet formulas of $x(s)$ are written by

$$\begin{aligned} x'(s) &= \alpha(s) \\ \alpha'(s) &= \kappa(s)x(s) - y(s) \\ y'(s) &= -\kappa(s)\alpha(s), \end{aligned} \tag{2.1}$$

where the function $\kappa(s)$ is called cone curvature function of the curve $x(s)$, $[17]$.

Let $x : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ with an arc length parameter s. Then
$x = x(s) = (x_1, x_2, x_3)$ can be given as

$$x(s) = \frac{f_s^{-1}}{2}(f^2 - 1, 2f, f^2 + 1) \tag{2.2}$$

for some non constant function $f(s)$ and $f_s = f'$, [18].

3. Smarandache Curves in The Null Cone $\mathbf{Q}^2$

In this part, we describe the Smarandache curves in accordance with the asymptotic orthonormal frame in $\mathbf{Q}^2$. Also, we obtain the asymptotic orthonormal frame and cone curvature function of the Smarandache partners lying on $\mathbf{Q}^2$ using cone frenet formulas.

Smarandache curve $\gamma = \gamma(s^*(s))$ of the curve x is a regular unit speed curve lying fully on $\mathbf{Q}^2$. Let $\{x, \alpha, y\}$ and $\{\gamma, \alpha_\gamma, y_\gamma\}$ be the moving asymptotic orthonormal frames of x and γ, respectively.

Definition 3.1. *Assume that x be unit speed spacelike curve lying on $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$. Then, $x\alpha-$smarandache curve of x is defined by*

$$\gamma_{x\alpha}(s^*) = \frac{c}{b}x(s) + \alpha(s), \tag{3.1}$$

where $c, b \in \mathbb{R}_0^+$.

Theorem 3.2. *Assume that x be unit speed spacelike curve in $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature $\kappa(s)$ and let $\gamma_{x\alpha}$ be $x\alpha-$smarandache curve with asymptotic orthonormal frame $\{\gamma_{x\alpha}, \alpha_{x\alpha}, y_{x\alpha}\}$. Then the following relations hold:*

i) The asymptotic orthonormal frame $\{\gamma_{x\alpha}, \alpha_{x\alpha}, y_{x\alpha}\}$ of the $x\alpha-$smarandache curve $\gamma_{x\alpha}$ is given as

$$\begin{bmatrix} \gamma_{x\alpha} \\ \alpha_{x\alpha} \\ y_{x\alpha} \end{bmatrix} = \begin{bmatrix} \frac{c}{b} & 1 & 0 \\ \psi & \frac{b}{c}\kappa\psi & -\frac{b}{c}\psi \\ \varrho_1 & \varrho_2 & \varrho_3 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ y \end{bmatrix}, \tag{3.2}$$

where

$$= \frac{c}{\sqrt{c^2 - 2\kappa b^2}}, \tag{3.3}$$

$$\Upsilon_1 = \frac{b}{c}(\frac{b}{c}\kappa' + \kappa)\psi\psi' + \frac{b^2}{c^2}\kappa(\psi')^2,$$

$$\Upsilon_2 = \frac{b}{c}\psi(\psi' + 2\frac{b}{c}\kappa\psi), \tag{3.4}$$

$$\Upsilon_3 = -\frac{b}{c}\psi(\frac{b}{c}{}' + \psi)$$

and

$$\varrho_1 = -(\Upsilon_1 + \frac{c}{2b}(2\Upsilon_1\Upsilon_3 + \Upsilon_2^2)) = -\Upsilon_1 + \frac{c}{b}\kappa_{\gamma_{x\alpha}}(s^*),$$

$$\varrho_2 = -(\Upsilon_2 + \frac{1}{2}(2\Upsilon_1\Upsilon_3 + \Upsilon_2^2)) = -\Upsilon_2 + \kappa_{\gamma_{x\alpha}}(s^*), \tag{3.5}$$

$$\varrho_3 = -\Upsilon_3.$$

ii) The cone curvature $\kappa_{\gamma_{x\alpha}}(s^*)$ of the curve $\gamma_{x\alpha}$ is given by

$$\kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2}(2\Upsilon_1\Upsilon_3 + \Upsilon_2^2), \tag{3.6}$$

where

$$s^* = \frac{1}{b} \int \sqrt{c^2 - 2b^2 \kappa(s)} ds.$$

Proof. **i)** We assume that the curve x is a unit speed spacelike curve with the asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature κ. If we differentiate the (3.1) with respect to s and taking into account (2.1), we have

$$\gamma'_{x\alpha}(s^*) = A\left(\alpha(s) + \frac{b}{c}\kappa x(s) - \frac{b}{c}y(s)\right), \tag{3.7}$$

where

$$\frac{ds^*}{ds} = \frac{1}{b}\sqrt{c^2 - 2b^2 \kappa(s)}, \tag{3.8}$$

$$= \frac{c}{\sqrt{c^2 - 2\kappa(s)b^2}}. \tag{3.9}$$

It can be readily observed that the tangent vector $\gamma'_{x\alpha}(s^*) = \alpha_{x\alpha}(s^*)$ is a unit spacelike vector.

Differentiating (3.7), we obtain equation as follows

$$\gamma''_{x\alpha}(s^*) = \Upsilon_1 x(s) + \Upsilon_2 \alpha(s) + \Upsilon_3 y(s), \tag{3.10}$$

where

$$\Upsilon_1 = \frac{b}{c}(\frac{b}{c}\kappa' + \kappa)\psi\psi' + \frac{b^2}{c^2}\kappa(\psi')^2,$$

$$\Upsilon_2 = \frac{b}{c}\psi(\psi' + 2\frac{b}{c}\kappa\psi),$$

$$\Upsilon_3 = -\frac{b}{c}\psi(\frac{b}{c}{}' + \psi).$$

$$y_{x\alpha}(s^*) = -\gamma''_{x\alpha} - \frac{1}{2}\langle\gamma''_{x\alpha}, \gamma''_{x\alpha}\rangle\gamma_{x\alpha}. \tag{3.11}$$

By the help of previous equation (3.11), we obtain

$$y_{x\alpha}(s^*) = \varrho_1 x(s) + \varrho_2 \alpha(s) + \varrho_3 y(s), \tag{3.12}$$

where $\varrho_1 = -(\Upsilon_1 + \frac{c}{2b}(2\Upsilon_1\Upsilon_3 + \Upsilon_2^2))$, $\varrho_2 = -(\Upsilon_2 + \frac{1}{2}(2\Upsilon_1\Upsilon_3 + \Upsilon_2^2))$, $\varrho_3 = -\Upsilon_3$.

ii) The curvature $\kappa_{\gamma_{x\alpha}}(s^*)$ of the $\gamma_{x\alpha}(s^*)$ is explicity obtained by

$$\kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2}\langle\gamma''_{x\alpha}, \gamma''_{x\alpha}\rangle$$

$$= -\frac{1}{2}\left(2\Upsilon_1\Upsilon_3 + \Upsilon_2^2\right). \tag{3.13}$$

Thus, the theorem is proved. $\square$

Definition 3.3. *Assume that x be unit speed spacelike curve lying on $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$. Then, $xy-$smarandache curve of x is defined by*

$$\gamma_{xy}(s^*) = \frac{1}{\sqrt{2cb}}\left(cx(s) + by(s)\right), \tag{3.14}$$

where $c, b \in \mathbb{R}_0^+$.

Theorem 3.4. *Assume that x be unit speed spacelike curve in $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature κ and let γ_{xy} be $xy-$smarandache curve with asymptotic orthonormal frame $\{\gamma_{xy}, \alpha_{xy}, y_{xy}\}$. Then the following relations hold:*

i) The asymptotic orthonormal frame $\{\gamma_{xy}, \alpha_{xy}, y_{xy}\}$ of the $xy-$smarandache curve γ_{xy} is given as

$$\begin{bmatrix} \gamma_{xy} \\ \alpha_{xy} \\ y_{xy} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{c}{2b}} & 0 & \sqrt{\frac{b}{2c}} \\ 0 & 1 & 0 \\ \frac{b\kappa^2\sqrt{2bc}}{(c-b\kappa)^2} & 0 & \frac{c\sqrt{2bc}}{(c-b\kappa)^2} \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ y \end{bmatrix}. \tag{3.15}$$

ii) The cone curvature $\kappa_{\gamma_{xy}}(s^*)$ of the curve γ_{xy} is given by

$$\kappa_{\gamma_{xy}}(s^*) = \frac{2bc\kappa(s)}{(c - b\kappa(s))^2}, \tag{3.16}$$

where

$$s^* = \frac{1}{\sqrt{2cb}} \int (c - b\kappa(s))\, ds. \tag{3.17}$$

Proof. **i)** We assume that the curve x is a unit speed spacelike curve with the asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature κ. Differentiating the equation (3.14) with respect to s and taking into account (2.1), we have

$$\gamma'_{xy}(s^*)\frac{ds^*}{ds} = \frac{1}{\sqrt{2cb}}(c - b\kappa(s))\,\overrightarrow{\alpha}(s). \tag{3.18}$$

By considering (3.17), we get

$$\gamma'_{xy}(s^*) = \alpha(s) = \alpha_{xy}. \tag{3.19}$$

Here, it can be readily observed that the tangent vector $\overrightarrow{\alpha}_{xy}$ is a unit spacelike vector.

$$\gamma''_{xy}(s^*)\frac{ds^*}{ds} = \kappa x(s) - y(s). \tag{3.20}$$

By substituting (3.17) into (3.20) and making necessary calculations, we obtain

$$\gamma''_{xy}(s^*) = \frac{\kappa\sqrt{2bc}}{(c - b\kappa)}\overrightarrow{x} - \frac{\sqrt{2bc}}{(c - b\kappa)^2}\overrightarrow{y}. \tag{3.21}$$

By the help of equation $y_{xy}(s^*) = -\gamma''_{xy} - \frac{1}{2}\left\langle \gamma''_{xy}, \gamma''_{xy} \right\rangle \gamma_{xy}$, we write

$$y_{xy}(s^*) = \frac{b\sqrt{2bc}.\kappa^2}{(c - b\kappa)^2}x(s) + \frac{c\sqrt{2bc}}{(c - b\kappa)^2}y(s). \tag{3.22}$$

ii) The curvature $\kappa_{\gamma_{xy}}(s^*)$ of the $\gamma_{xy}(s^*)$ is explicity obtained by

$$\kappa_{\gamma_{xy}}(s^*) = \frac{-\left\langle \gamma''_{xy}, \gamma''_{xy} \right\rangle}{2} = \frac{2bc\kappa(s)}{(c - b\kappa(s))^2}.$$

$\square$

Definition 3.5. *Assume that x be unit speed spacelike curve lying on $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$. At that time, $\alpha y-$smarandache curve of x is defined by*

$$\gamma_{\alpha y}(s^*) = \alpha(s) + \frac{b}{c}y(s), \tag{3.23}$$

where $c, b \in \mathbb{R}_0^+$.

Theorem 3.6. *Assume that x be unit speed spacelike curve in $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature κ and let $\gamma_{\alpha y}$ be $\alpha y-$smarandache curve with asymptotic orthonormal frame $\{\gamma_{\alpha y}, \alpha_{\alpha y}, y_{\alpha y}\}$. At the time the following relations hold:*

i) The asymptotic orthonormal frame $\{\gamma_{\alpha y}, \alpha_{\alpha y}, y_{\alpha y}\}$ of the $\alpha y-$smarandache curve $\gamma_{\alpha y}$ is given as

$$\begin{bmatrix} \gamma_{\alpha y} \\ \alpha_{\alpha y} \\ y_{\alpha y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{b}{c} \\ \frac{c\sqrt{\kappa}}{\sqrt{b^2-2c^2}} & \frac{b\sqrt{\kappa}}{\sqrt{b^2-2c^2}} & \frac{c\sqrt{\kappa}}{\kappa\sqrt{b^2-2c^2}} \\ \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ y \end{bmatrix}, \tag{3.24}$$

where

$$\zeta_1 = \frac{c\kappa'}{b^2 - 2c^2}\left(\frac{c - 2\kappa b}{2\kappa}\right),$$

$$\zeta_2 = \frac{c\kappa'}{b^2 - 2c^2}\left(\frac{c - b - 1}{2\kappa}\right), \tag{3.25}$$

$$\zeta_3 = \frac{c\kappa'}{b^2 - 2c^2}\left(\frac{b\kappa - c}{2\kappa^2}\right)$$

and

$$\omega_1 = -\zeta_1,$$

$$\omega_2 = -(\zeta_2 + \frac{1}{2}\left(2\zeta_1\zeta_3 + \zeta_2^2\right)), \tag{3.26}$$

$$\omega_3 = -(\zeta_3 + \frac{b}{2c}\left(2\zeta_1\zeta_3 + \zeta_2^2\right)).$$

ii) The cone curvature $\kappa_{\gamma_{\alpha y}}(s^*)$ of the curve $\gamma_{\alpha y}$ is given by

$$\kappa_{\gamma_{\alpha y}}(s^*) = \frac{-c^2}{8\left(b^2 - 2c^2\right)^2}(\frac{\kappa'}{\kappa})^2\left(\frac{(c - 2\kappa b)(b\kappa - c)}{\kappa} + (c - b - 1)^2\right), \tag{3.27}$$

where

$$s^* = \frac{\sqrt{b^2 - 2c^2}}{c}\int\sqrt{\kappa(s)}ds. \tag{3.28}$$

Proof. **i)** Let the curve x be a unit speed spacelike curve with the asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature κ. Differentiating the equation (3.23) with respect to s and taking into account (2.1), we find

$$\gamma'_{\alpha y}(s^*)\frac{ds^*}{ds} = \kappa\overrightarrow{x(s)} - \frac{b}{c}\kappa\overrightarrow{\alpha(s)} - \overrightarrow{y(s)}.$$

This can be written as following

$$\alpha_{\alpha y}(s^*)\frac{ds^*}{ds} = \kappa\overrightarrow{x(s)} - \frac{b}{c}\kappa\overrightarrow{\alpha(s)} - \overrightarrow{y(s)}, \tag{3.29}$$

where

$$\frac{ds^*}{ds} = \frac{\sqrt{b^2 - 2c^2}}{c}\sqrt{\kappa(s)}. \tag{3.30}$$

By substituting (3.30) into (3.29), we find

$$\alpha_{\alpha y}(s^*) = \frac{c\sqrt{\kappa}}{\sqrt{b^2 - 2c^2}}\overrightarrow{x} - \frac{b\sqrt{\kappa}}{\sqrt{b^2 - 2c^2}}\overrightarrow{\alpha} - \frac{c\sqrt{\kappa}}{\kappa\sqrt{b^2 - 2c^2}}\overrightarrow{y}. \tag{3.31}$$

Differentiating (3.31) and using (3.30), we get

$$\gamma''_{\alpha y}(s^*) = \zeta_1 x(s) + \zeta_2 \alpha(s) + \zeta_3 y(s),$$

where $\zeta_1 = \frac{c\kappa'}{b^2 - 2c^2}\left(\frac{c - 2\kappa b}{2\kappa}\right), \zeta_2 = \frac{c\kappa'}{b^2 - 2c^2}\left(\frac{c - b - 1}{2\kappa}\right), \zeta_3 = \frac{c\kappa'}{b^2 - 2c^2}\left(\frac{b\kappa - c}{2\kappa^2}\right).$

$$y_{\alpha y}(s^*) = -\gamma''_{\alpha y} - \frac{1}{2}\left\langle\gamma''_{\alpha y}, \gamma''_{\alpha y}\right\rangle\gamma_{\alpha y}. \tag{3.32}$$

By the help of equation (3.32), we obtain

$$y_{\alpha y}(s^*) = \omega_1 x(s) + \omega_2 \alpha(s) + \omega_3 y(s), \tag{3.33}$$

where $\omega_1 = -\zeta_1, \omega_2 = -(\zeta_2 + \frac{1}{2}\left(2\zeta_1\zeta_3 + \zeta_2^2\right)), \omega_3 = -(\zeta_3 + \frac{b}{2c}\left(2\zeta_1\zeta_3 + \zeta_2^2\right)).$

ii) The curvature $\kappa_{\gamma_{\alpha y}}(s^*)$ of the $\gamma_{\alpha y}(s^*)$ is explicity obtained by

$$\kappa_{\gamma_{\alpha y}}(s^*) = -\frac{c^2}{8\left(b^2 - 2c^2\right)^2}\left(\frac{\kappa'}{\kappa}\right)^2\left(\frac{(c - 2\kappa b)(b\kappa - c)}{\kappa} + (c - b - 1)^2\right).$$

$\square$

Definition 3.7. *Assume that x be unit speed spacelike curve lying on $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$. Then, $x\alpha y-$smarandache curve of x is defined by*

$$\gamma_{x\alpha y}(s^*) = \frac{1}{\sqrt{2cc^* + b^2}}\left(cx(s) + b\alpha(s) + c^*y(s)\right), \tag{3.34}$$

where $c, c^*, b \in \mathbb{R}_0^+$.

Theorem 3.8. *Assume that x be unit speed spacelike curve in $\mathbf{Q}^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature κ and let $\gamma_{x\alpha y}$ be $x\alpha y-$smarandache curve with asymptotic orthonormal frame $\{\gamma_{x\alpha y}, \alpha_{x\alpha y}, y_{x\alpha y}\}$. Then the following relations hold:*

i) The asymptotic orthonormal frame $\{\gamma_{x\alpha y}, \alpha_{x\alpha y}, y_{x\alpha y}\}$ of the $x\alpha y-$smarandache curve $\gamma_{x\alpha y}$ is given as

$$\begin{bmatrix}\gamma_{x\alpha y}\\\alpha_{x\alpha y}\\y_{x\alpha y}\end{bmatrix} = \begin{bmatrix}\frac{c}{\sqrt{2cc^* + b^2}} & \frac{b}{\sqrt{2cc^* + b^2}} & \frac{c^*}{\sqrt{2cc^* + b^2}}\\\rho_1 & \rho_2 & \rho_3\\\sigma_1 & \sigma_2 & \sigma_3\end{bmatrix}\begin{bmatrix}x\\\alpha\\y\end{bmatrix} \tag{3.35}$$

where

$$\eta = \sqrt{(c - c^*\kappa(s))^2 - 2b^2\kappa(s)};$$

$$\rho_1 = \frac{b\kappa(s)}{\eta}, \rho_2 = \frac{c - c^*\kappa(s)}{\eta}, \rho_3 = -\frac{b}{\eta}; \tag{3.36}$$

$$\xi_1 = (\rho_1' + \rho_2\kappa), \xi_2 = \rho_2' + \rho_1 + \rho_3\kappa, \xi_3 = -\rho_3' - \rho_2$$

and

$$\sigma_1 = -\xi_1 - \frac{c}{2\sqrt{2cc^* + b^2}}\left(2\xi_1\xi_3 + \xi_2^2\right),$$

$$\sigma_2 = -\xi_2 - \frac{b}{2\sqrt{2cc^* + b^2}}\left(2\xi_1\xi_3 + \xi_2^2\right), \tag{3.37}$$

$$\sigma_3 = -\xi_3 - \frac{c^*}{2\sqrt{2cc^* + b^2}}\left(2\xi_1\xi_3 + \xi_2^2\right).$$

ii) *The cone curvature $\kappa_{\gamma_{x\alpha y}}(s^*)$ of the curve $\gamma_{x\alpha y}$ is given by*

$$\kappa_{\gamma_{yx\alpha}}(s^*) = \left(b\left(\frac{\kappa}{\eta}\right)' + \frac{c - c^*\kappa}{\eta}\kappa\right)\left(\left(\frac{b}{\eta}\right)' + \frac{c - c^*\kappa}{\eta}\right)$$

$$- \frac{1}{2}\left(\frac{c - c^*\kappa}{\eta}\right)', \tag{3.38}$$

where

$$s^* = \frac{1}{\sqrt{2cc^* + b^2}}\int\sqrt{(c - c^*\kappa(s))^2 - 2b^2\kappa(s)}ds, \ b, c, c^* \in \mathbb{R}_0^+. \tag{3.39}$$

Proof. **i)** Differentiating the equation (3.34) with respect to s and taking into account (2.1), we find

$$\gamma_{x\alpha y}'(s^*)\frac{ds^*}{ds} = \frac{1}{\sqrt{2cc^* + b^2}}\left(b\kappa\overrightarrow{x(s)} + (c - c^*\kappa)\overrightarrow{\alpha(s)} - \overrightarrow{by(s)}\right). \tag{3.40}$$

This can be written as follows

$$\alpha_{x\alpha y}(s^*) = \frac{b\kappa}{\eta}\overrightarrow{x(s)} + \frac{c - c^*\kappa}{\eta}\overrightarrow{\alpha(s)} - \frac{b}{\eta}\overrightarrow{y(s)} \tag{3.41}$$

or

$$\alpha_{x\alpha y}(s^*) = \rho_1\overrightarrow{x(s)} + \rho_2\overrightarrow{\alpha(s)} - \rho_3\overrightarrow{y(s)}, \tag{3.42}$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2cc^* + b^2}}\sqrt{(c - c^*\kappa)^2 - 2b^2\kappa}. \tag{3.43}$$

Differentiating (3.42) and using (3.43), we get

$$\gamma_{xy\alpha}''(s^*) = \xi_1 x(s) + \xi_2\alpha(s) + \xi_3 y(s),$$

where $\xi_1 = (\rho_1' + \rho_2\kappa), \xi_2 = \rho_2' + \rho_1 + \rho_3\kappa, \xi_3 = -\rho_3' - \rho_2$.

$$y_{x\alpha y}(s^*) = -\gamma_{x\alpha y}'' - \frac{1}{2}\left\langle\gamma_{x\alpha y}'', \gamma_{x\alpha y}''\right\rangle\gamma_{x\alpha y}. \tag{3.44}$$

By the help of equation (3.44), we obtain

$$y_{xay}(s^*) = \sigma_1 x(s) + \sigma_2 \alpha(s) + \sigma_3 y(s), \tag{3.45}$$

where $\sigma_1 = -\xi_1 - \frac{c}{2\sqrt{2cc^*+b^2}}\left(2\xi_1\xi_3 + \xi_2^2\right), \sigma_2 = -\xi_2 - \frac{b}{2\sqrt{2cc^*+b^2}}\left(2\xi_1\xi_3 + \xi_2^2\right), \sigma_3 = -\xi_3 - \frac{c^*}{2\sqrt{2cc^*+b^2}}\left(2\xi_1\xi_3 + \xi_2^2\right).$

ii) From $\kappa_{\gamma_{xay}}(s^*) = -\frac{1}{2}\left\langle \gamma''_{xay}, \gamma''_{xay}\right\rangle$, we have

$$\kappa_{\gamma_{yxa}}(s^*) = \left(b\left(\frac{\kappa}{\eta}\right)' + \frac{c-c^*\kappa}{\eta}\kappa\right)\left(\left(\frac{b}{\eta}\right)' + \frac{c-c^*\kappa}{\eta}\right) - \frac{1}{2}\left(\frac{c-c^*\kappa}{\eta}\right)'.$$

$\square$

Theorem 3.9. *Let $x : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ as follows*

$$x(s) = \frac{f_s^{-1}}{2}(f^2 - 1, 2f, f^2 + 1), \tag{3.46}$$

for some non constant function $f(s)$. Then we can write the following conditions:

1) If x is a $x\alpha-$smarandache curve, then the $x\alpha-$smarandache curve $\gamma_{x\alpha}$ can be written as

$$\gamma_{x\alpha}(s^*) = (\frac{c}{b} - f_s^{-1}f_{ss})x(s) + (f, 1, f). \tag{3.47}$$

2) If x is a $xy-$smarandache curve, then the $xy-$smarandache curve γ_{xy} can be written as

$$\gamma_{xy}(s^*) = \frac{1}{\sqrt{2bc}}\left(\begin{array}{c}(c - \frac{1}{2}f_s^{-2}f_{ss}^2)x(s) + f_s^{-1}f_{ss}(f, 1, f)\\-f_s(1.0, 1)\end{array}\right). \tag{3.48}$$

3) If x is a $\alpha y-$smarandache curve, then the $\alpha y-$smarandache curve $\gamma_{\alpha y}$ can be written as

$$\gamma_{\alpha y}(s^*) = \left(\begin{array}{c}\left(-f_s^{-1}f_{ss} - \frac{b}{2c}f_s^{-2}f_{ss}^2\right)x(s) + \left(1 + \frac{b}{c}f_s^{-1}f_{ss}\right)(f, 1, f)\\-\frac{b}{c}f_s(1.0, 1)\end{array}\right). \tag{3.49}$$

4) If x is a $x\alpha y-$smarandache curve, then the $x\alpha y-$smarandache curve $\gamma_{x\alpha y}$ can be written as

$$\gamma_{x\alpha y}(s^*) = \frac{1}{\sqrt{2cc^* + b^2}}\left(\begin{array}{c}\left(c - bf_s^{-1}f_{ss} - \frac{c^*}{2}f_s^{-2}f_{ss}^2\right)x(s)\\+\left(b + c^*f_s^{-1}f_{ss}\right)(f, 1, f) - f_s(1.0, 1)\end{array}\right), \tag{3.50}$$

where $c, b, c^* \in \mathbb{R}_0^+$.

Proof. It is obvious from (3.1), (3.14), (3.23), (3.34) and (3.46). $\square$

We can give the following example to hold special Smarandache curves in the null cone $\mathbf{Q}^2$. Special $x\alpha$, xy, αy, and $x\alpha y-$smarandache curves of x curves are given in Figure 1 A, C, E, G, I, respectively. These figures rotated in three dimensions are also given in Figure 1 B, D, F, H, J, respectively.

Example 3.10. *The curve*

$$x(s) = \left(\frac{\cosh s}{2} - \frac{1}{\cosh s}, \tanh s, \frac{\cosh s}{2}\right)$$

is spacelike in $\mathbf{Q}^2$ with arc length parameter s. Also, the shape of the x curve is given as follows:

Then we can write the smarandache curves of the x curve as follows:

i) $x\alpha-$smarandache curve $\gamma_{x\alpha}$ is given by

$$\gamma_{x\alpha}(s) = \begin{pmatrix} d\left(\frac{\cosh s}{2} - \frac{1}{\cosh s}\right) + \frac{\sinh s}{2} - \frac{\tanh s}{\cosh s}, \\ d\tanh s + \frac{1}{\cosh^2 s}, \\ d\frac{\cosh s}{2} + \frac{\sinh s}{2} \end{pmatrix}$$

ii) $xy-$smarandache curve γ_{xy} is given by

$$\gamma_{xy}(s) = \begin{pmatrix} m\cosh s - n\tanh s\sinh s - d\left(\frac{1c+\tanh^2 s}{\cosh s}\right), \\ c\tanh s\left(e - \frac{\tanh^2 s}{2}\right), \\ d\left(\left(c - \frac{\tanh^2 s}{2}\right)\frac{\cosh s}{2} - \frac{1}{\cosh s}\right) \end{pmatrix}$$

iii) $\alpha y-$ smarandache curve $\gamma_{\alpha y}$ is given by

$$\gamma_{\alpha y}(s) = \begin{pmatrix} \begin{pmatrix} \left(\frac{\cosh s}{2} - \frac{1}{\cosh s}\right)(1 - e\tanh s)\tanh s \\ + \sinh s\left(1 + d\tanh s\right) - d\cosh s \end{pmatrix}, \\ -\tanh^2 s\left(1 + d\tanh s\right) + d\tanh s, \\ \frac{\sinh s}{2} - d\cosh s - e\sinh s\tanh s \end{pmatrix}$$

iv) $x\alpha y-$smarandache curve $\gamma_{x\alpha y}$ is given by

$$\gamma_{x\alpha y}(s) = \begin{pmatrix} \begin{pmatrix} m\sinh s - n\cosh s + d\sinh s\tanh s \\ -\frac{c}{\cosh s} - \frac{b\sinh s - c^*\tanh^2 s}{\cosh^2 s} \end{pmatrix}, \\ m + n\tanh s + k\tanh^2 s + l\tanh^3 s, \\ m\cosh s + d\sinh s + c\tanh s\sinh s, \end{pmatrix}$$

where $m, n, d, c, k, l, e \in \mathbb{R}_0^+$.

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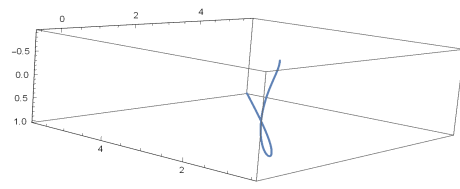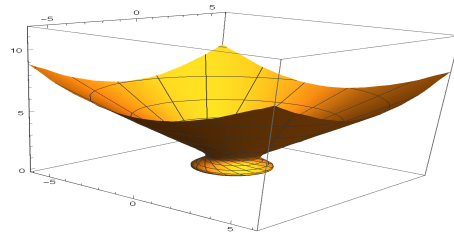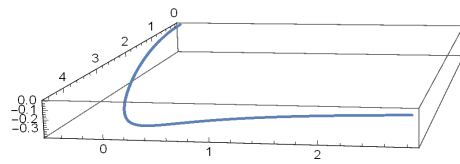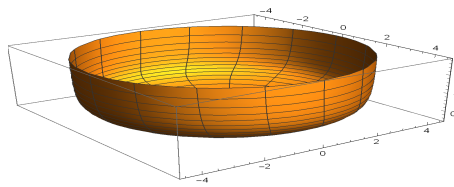
(A) The curve x
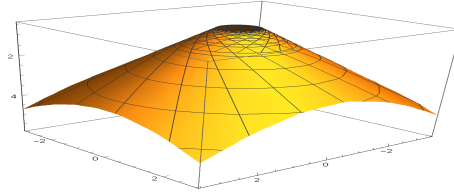
(B) The rotated surface of curve x
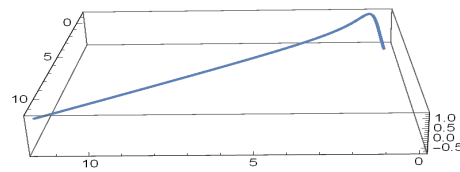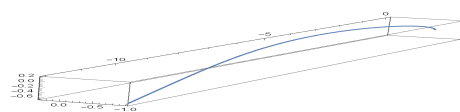
(C) $x\alpha-$smarandache curve
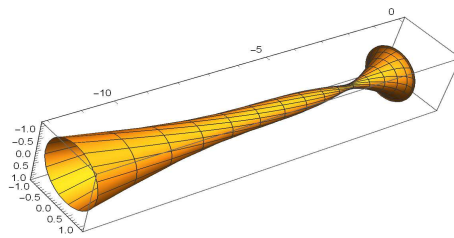
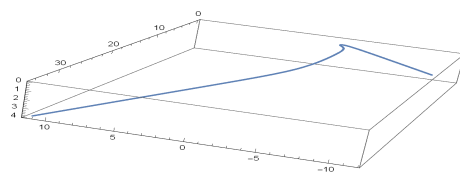(D) $x\alpha$- smarandache surface.

(E) $xy-$smarandache curve

(F) $xy-$smarandache surface.
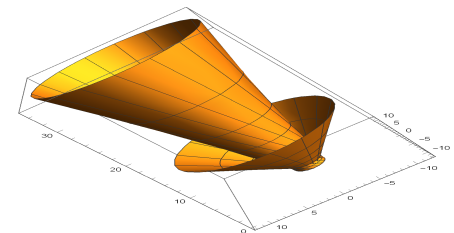
(G) $\alpha y-$smarandache curve.

(H) $\alpha y-$smarandache surface.

(I) $x\alpha y-$smarandache curve.

(J) $x\alpha y-$smarandache surface.

FIGURE 1. Graphics of smarandache surfaces and smarandache curves

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