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Generalized Family of Approximating Schemes based on Newton Interpolating Polynomials and its Error Analysis

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Abstract. A generalized family of $(2\jmath + 2)$-point n-ary approximating subdivision schemes by using Newton interpolating polynomials is presented for the generation of curves and surfaces, where $\jmath \geq 0$ and $n \geq 2$. This family unifies some well known curve and surface schemes. Furthermore, the error bound evaluation technique for the proposed schemes is also discussed.

AMS (MOS) Subject Classification Codes: 65D17; 65D07; 65D05
Key Words: Approximating subdivision scheme; subdivision curves and surfaces; Newton polynomial; unified schemes; error bound

1. Introduction

Subdivision schemes have been a well-liked technique in CAGD (Computer Aided Geometric Design) to produce curves and surfaces. Subdivision schemes have various applications in graphics, image processing, engineering etc and thus has become an important area of study. A subdivision process refines the initial polygon recursively to a set of refined polygons which converge to a smooth limiting curve/surface. New points are added at each refinement level into the existing polygon while the old points may remain intact or vanish in all subsequent sequences of control polygons. Furthermore, arity of the scheme refers to how many points are fitted at new level $\kappa + 1$ between each pair of adjacent points from level κ, i.e, when two points are fitted then the resulting scheme is called binary, when three points are fitted then the resulting scheme is called ternary and when n points are fitted, the resulting scheme is called n-ary subdivision scheme.

In the preceding years, vast amount of work has been done in generating subdivision curves. A significant assessment of different approximating subdivision scheme can be found in [1, 4, 11, 19, 20, 21]. Subsequently, subdivision for surface design is also creditable. It is a modified form of subdivision curve and gained more popularity, most notably in the computer animation industry. The idea of subdivision curve to subdivision surface was initially given by [3]. Also, [6] extended Chaikin's corner-cutting method successfully for curves to surfaces.

Note that the formulation of existing approximating subdivision schemes for curve/surface is different to each other. Several subdivision schemes including triangular mesh schemes [8, 12, 16, 17], quad mesh schemes [3, 7, 13, 18, 27, 30] and combined triangular-quad (tri-quad) mesh schemes [26, 29] have been proposed. These schemes are the result of modifications, convex combination of two subdivision schemes, polynomial interpolation, convolvement and tensor product of the mask of present schemes. So, it is interesting to present a general formula, which covers most of the existing schemes and constructing methodologies, which is independent of all the previous methods. So, it is new class of construction of subdivision schemes. Least squares based subdivision schemes are introduced in [2, 24, 25]. In this paper, we present a general formula for curve designing, which is generalization of existing well-known schemes, then the work is extended for regular quad meshes. Formulation of both formulae for curves and surfaces is the result of using one and two dimensional Newton's interpolation formulae.

The paper is planned since go after: Section 2 gives a brief fundamental facts of this paper. Section 3 presents a new generalized form of $(2\jmath + 2)$-point approximating scheme for subdivision curves and Section 4 is devoted for tensor product scheme for subdivision surfaces. In Section 5, analysis and applications of proposed schemes are discussed. Finally, Section 6 consists of the error bounds for the proposed approximating subdivision scheme.

2. PRELIMINARIES

The n-ary approximating subdivision scheme for univariate case, maps an initial polygon $p^\kappa = \{p_i^\kappa\}_{i\in\mathbb{Z}}$ to a new refined polygon $p^{\kappa+1} = \{p_i^{\kappa+1}\}_{i\in\mathbb{Z}}$, is defined by

$$p_{ni+\lambda}^{\kappa+1} = \sum_{i\in\mathbb{Z}} a_{\lambda,j} p_{i+j}^{\kappa}, \quad \lambda = 0, 1, 2, \cdots, n-1, \tag{2. 1}$$

where $\{a_i\}_{i\in\mathbb{Z}}$ is the set of masks of the subdivision scheme. The n-ary subdivision scheme converges uniformly if

$$\sum_{j\in\mathbb{Z}} a_{\lambda,j} = 1, \quad \lambda = 0, 1, 2, \cdots, n-1. \tag{2. 2}$$

Let $N_j(t)$ be the fundamental Newton polynomial corresponding to the node $\{j\}_{-\jmath}^{\jmath+1}$ defined by

$$P_{2\jmath+1}(t) = \sum_{j=-\jmath}^{\jmath+1} \tilde{a}_j N_j(t), \tag{2. 3}$$

where $P_{2\jmath+1} \in \coprod_{2\jmath+1}$(space of polynomials of degree $\leq 2\jmath+1$).
Here, $\tilde{a}_j = y[t_0, \cdots, t_n]$ are the divided differences and are symmetric in nature. These can be evaluated by the following method.

$$\tilde{a}_j = y[t_{-\jmath}, \cdots, t_j] = \begin{cases} y_{-\jmath} & t_\jmath = t_j \\ \sum\limits_{i=-\jmath}^{j} y_i \left(\prod\limits_{i=-\jmath,\ i\neq\kappa}^{j} (i-\kappa) \right)^{-1} & t_\jmath \neq t_j, \end{cases} \tag{2. 4}$$

and $N_j(t)$ can be originated by the subsequent way,

$$N_j(t) = \prod_{\kappa=-\jmath}^{j-1} (t-\kappa) = \begin{cases} 1 & \kappa = j \\ \prod\limits_{\kappa=-\jmath}^{j-1} (t-\kappa) & \kappa \neq j. \end{cases} \tag{2. 5}$$

3. Generalized approximating schemes

In this section, we will construct $(2\jmath+2)$-point binary and ternary approximating subdivision schemes for curve and then propose a generalized family of $(2\jmath+2)$-point n-ary approximating subdivision schemes.

3.1. $(2\jmath+2)$**-point binary approximating scheme.** Chaikin's 2-point binary approximating subdivision scheme [4] can be obtained by setting $\jmath = 0$ and $n = 2$ in (2. 3) and evaluating the Newton's polynomial at $t = \frac{1}{4}, \frac{3}{4}$. Thus (2. 3) takes the form

$$P_1(t) = \sum_{j=0}^{1} \tilde{a}_j N_j\,(t)\,, \tag{3. 6}$$

where the node points of the Newton polynomials are $N_0(t)$ and $N_1(t)$ which reproduce the linear polynomial as follows

$$P_1(t) = \sum_{\mu=0}^{1} (-1)^{\mu} \left[\sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} p_\nu \right] \Gamma(t,\mu), \tag{3. 7}$$

where $\Gamma(t,\mu)$ is the Gamma function defined as

$$\Gamma(t,\mu) = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)\Gamma(\mu+1)}.$$

Now, we have the following proposed 2-point binary subdivision scheme

$$p_{2i}^{1} = P_1\left(i+\frac{3}{4}\right), \qquad p_{2i+1}^{1} = P_1\left(i+\frac{1}{4}\right),$$

Here, we consider the case for $i = 0$ and subdivision level $\kappa = 0$ as it is adequate to construct stationary and uniform subdivision schemes reproducing polynomials of a fixed degree as follows

$$\begin{aligned} P_1\left(\frac{3}{4}\right) &= \frac{3}{4}p_0 + \frac{1}{4}p_1, \\ P_1\left(\frac{1}{4}\right) &= \frac{1}{4}p_0 + \frac{3}{4}p_1. \end{aligned}$$

Now, suppose that at next level of subdivision(i.e, $(\kappa+1)$-th level), the point $p^{\kappa+1}_{2i+\lambda}$, which is the affine combination of two points $p^{\kappa+1}_{i}$ and $p^{\kappa+1}_{i+1}$ at the same level, is attached to the constraint value $\frac{2i+\lambda}{2^{\kappa+1}}$. Thus, we get Chaikin's 2-point binary scheme [4] as follows

$$\begin{cases} p^{\kappa+1}_{2i} = \frac{3}{4}p^{\kappa}_{i} + \frac{1}{4}p^{\kappa}_{i+1} \\ p^{\kappa+1}_{2i+1} = \frac{1}{4}p^{\kappa}_{i} + \frac{3}{4}p^{\kappa}_{i+1}. \end{cases} \tag{3. 8}$$

The scheme (3. 8) can be written as

$$p^{\kappa+1}_{2i+\lambda} = \sum_{\mu=0}^{1}(-1)^{\mu}\left[\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}p^{\kappa}_{i+\nu}\right]\Gamma(t,\mu), \qquad \lambda = 0,1, \tag{3. 9}$$

where

$$\Gamma(t,\mu) = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)\Gamma(\mu+1)}, \qquad t = \frac{2\lambda+1}{4}.$$

Similarly, Dyn's 4-point binary approximating subdivision scheme [9] is obtained by setting $\jmath = 1$ and $n = 2$ in (2. 3), where the node points of Newton polynomial are $\{N_j(t)\}^{2}_{-1}$, which can be written as

$$p^{\kappa+1}_{2i+\lambda} = \sum_{\mu=0}^{3}(-1)^{\mu}\left[\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}p^{\kappa}_{i+(\nu-1)}\right]\Gamma(t+1,\mu), \qquad \lambda = 0,1, \tag{3. 10}$$

where

$$\Gamma(t+1,\mu) = \frac{\Gamma(t+2)}{\Gamma(t+2-\mu)\Gamma(\mu+1)}, \qquad t = \frac{2\lambda+1}{4}.$$

Similarly, we get binary 6-point scheme by setting $\jmath = 2$ in (2. 3) as follows

$$p^{\kappa+1}_{2i+\lambda} = \sum_{\mu=0}^{5}(-1)^{\mu}\left[\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}p^{\kappa}_{i+(\nu-2)}\right]\Gamma(t+2,\mu), \qquad \lambda = 0,1. \tag{3. 11}$$

where

$$\Gamma(t+2,\mu) = \frac{\Gamma(t+3)}{\Gamma(t+3-\mu)\Gamma(\mu+1)}, \qquad t = \frac{2\lambda+1}{4}.$$

Therefore, using (3. 9)-(3. 11), the general form of $(2\jmath+2)$-point binary approximating scheme is proposed as follows

$$p^{\kappa+1}_{2i+\lambda} = \sum_{\mu=0}^{2\jmath+1}(-1)^{\mu}\left[\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}p^{\kappa}_{i+(\nu-\jmath)}\right]\Gamma(t+\jmath,\mu), \tag{3. 12}$$

where $\Gamma(t+\jmath,\mu)=\frac{\Gamma(t+\jmath+1)}{\Gamma(t+\jmath+1-\mu)\Gamma(\mu+1)}$, $\lambda=0,1$ corresponding to $t=\frac{2\lambda+1}{4}$, $\jmath\geq 0$ and subdivision level $\kappa\geq 0$.

3.2. **$(2\jmath+2)$-point ternary approximating scheme.** A 2-point ternary approximating subdivision scheme can be obtained by setting $\jmath=0$ and $n=3$ in (2. 3) and considering the Newton's polynomial $\{N_j(t)\}$ at node points $\{0,1\}$ as follows

$$P_1(t)=\sum_{j=0}^{1}\tilde{a}_jN_j\left(t\right). \tag{3. 13}$$

Now by using (2. 4) and (2. 5) in (3. 13), we have

$$P_1(t) \quad = \quad p_0+\left(p_1-p_0\right)(t). \tag{3. 14}$$

For uniform and stationary ternary 2-point approximating scheme, consider the following

$$p^1_{3i}=P_1\left(i+\frac{1}{6}\right),\qquad p^1_{3i+1}=P_1\left(i+\frac{1}{2}\right),\qquad p^1_{3i+2}=P_1\left(i+\frac{5}{6}\right). \tag{3. 15}$$

From (3. 14), we have

$$\begin{aligned} P_1\left(\frac{1}{6}\right) &= \frac{5}{6}p_0+\frac{1}{6}p_1,\\ P_1\left(\frac{1}{2}\right) &= \frac{1}{2}p_0+\frac{1}{2}p_1,\\ P_1\left(\frac{5}{6}\right) &= \frac{1}{6}p_0+\frac{5}{6}p_1, \end{aligned}$$

which yield the following iterative form of ternary 2-point approximating subdivision scheme

$$\begin{cases} p^{\kappa+1}_{3i}=\frac{5}{6}p^{\kappa}_i+\frac{1}{6}p^{\kappa}_{i+1},\\ p^{\kappa+1}_{3i+1}=\frac{1}{2}p^{\kappa}_i+\frac{1}{2}p^{\kappa}_{i+1},\\ p^{\kappa+1}_{3i+2}=\frac{1}{6}p^{\kappa}_i+\frac{5}{6}p^{\kappa}_{i+1}. \end{cases} \tag{3. 16}$$

Above expression can be formed as following

$$p^{\kappa+1}_{3i+\lambda}=\sum_{\mu=0}^{1}(-1)^{\mu}\left[\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}p^{\kappa}_{i+\nu}\right]\Gamma(t,\mu),\qquad \lambda=0,1,2, \tag{3. 17}$$

where

$$\Gamma(t,\mu)=\frac{\Gamma(t+1)}{\Gamma(t+1-\mu)\Gamma(\mu+1)},\qquad t=\frac{2\lambda+1}{6}.$$

Similarly, 4-point ternary approximating subdivision scheme is obtained by setting $\jmath=1$ and $n=3$ in (2. 3). The node points of Newton polynomial are $\{N_j(t)\}^{2}_{-1}$ which reproduce cubic polynomial using (2. 4) and (2. 5) as follows

$$\begin{aligned} P_3(t) &= \sum_{j=-1}^{2} \tilde{a}_j N_j(t). \\ &= p_{-1} + (p_0 - p_{-1})(t+1) + \frac{1}{2}(p_1 - 2p_0 + p_{-1})(t^2+t) \\ &\quad + \frac{1}{6}(p_2 - 3p_1 + 3p_0 - p_{-1})(t^3 - t). \end{aligned} \tag{3. 18}$$

For uniform and stationary ternary 4-point approximating scheme, consider the following

$$p^1_{3i} = P_1\left(i + \frac{1}{6}\right), \qquad p^1_{3i+1} = P_1\left(i + \frac{1}{2}\right), \qquad p^1_{3i+2} = P_1\left(i + \frac{5}{6}\right). \tag{3. 19}$$

From (3. 18), we get

$$\begin{aligned} P_3\left(\frac{1}{6}\right) &= -\frac{55}{1296}p_{-1} + \frac{385}{432}p_0 + \frac{77}{432}p_1 - \frac{35}{1296}p_2 \\ P_3\left(\frac{1}{2}\right) &= -\frac{1}{16}p_1 + \frac{9}{16}p_0 + \frac{9}{16}p_1 - \frac{1}{16}p_2 \\ P_3\left(\frac{5}{6}\right) &= -\frac{35}{1296}p_{-1} + \frac{77}{432}p_0 + \frac{385}{432}p_1 - \frac{55}{1296}p_2. \end{aligned}$$

This implies the following iterative form of ternary 4-point approximating subdivision scheme

$$\begin{cases} p^{\kappa+1}_{3i} = -\frac{55}{1296}p^{\kappa}_{i-1} + \frac{385}{432}p^{\kappa}_{i} + \frac{77}{432}p^{\kappa}_{i+1} - \frac{35}{1296}p^{\kappa}_{i+2}, \\ p^{\kappa+1}_{3i+1} = -\frac{1}{16}p^{\kappa}_{i-1} + \frac{9}{16}p^{\kappa}_{i} + \frac{9}{16}p^{\kappa}_{i+1} - \frac{1}{16}p^{\kappa}_{i+2}, \\ p^{\kappa+1}_{3i+2} = -\frac{35}{1296}p^{\kappa}_{i-1} + \frac{77}{432}p^{\kappa}_{i} + \frac{385}{432}p^{\kappa}_{i+1} - \frac{55}{1296}p^{\kappa}_{i+2}. \end{cases} \tag{3. 20}$$

Briefly, we can write (3. 20) as

$$p^{\kappa+1}_{3i+\lambda} = \sum_{\mu=0}^{3}(-1)^{\mu}\left[\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}p^{\kappa}_{i+(\nu-1)}\right]\Gamma(t+1,\mu), \qquad \lambda = 0,1,2 \tag{3. 21}$$

where

$$\Gamma(t+1,\mu) = \frac{\Gamma(t+2)}{\Gamma(t+2-\mu)\Gamma(\mu+1)}, \qquad t = \frac{2\lambda+1}{6}.$$

Therefore, using (3. 17) and (3. 21), the general form of $(2\jmath+2)$-point ternary approximating scheme is proposed as follows

$$p^{\kappa+1}_{3i+\lambda} = \sum_{\mu=0}^{2\jmath+1}(-1)^{\mu}\left[\sum_{\nu=0}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}p^{\kappa}_{i+(\nu-\jmath)}\right]\Gamma(t+\jmath,\mu), \tag{3. 22}$$

where $\Gamma(t+\jmath,\mu) = \frac{\Gamma(t+\jmath+1)}{\Gamma(t+\jmath+1-\mu)\Gamma(\mu+1)}$, $\lambda = 0,1,2$ corresponding to $t = \frac{2\lambda+1}{6}$, $\jmath \geq 0$ and subdivision level $\kappa \geq 0$.

3.3. **Generalized $(2\jmath+2)$-point n-ary approximating scheme.** Here, the generalized form of $(2\jmath+2)$-point n-ary approximating subdivision scheme by using Newton interpolating polynomial is presented. This generalized form will be helpful in obtaining subdivision rules more rapidly. The generalized form of $(2\jmath+2)$-point n-ary approximating subdivision scheme is given as follows.

$$p^{\kappa+1}_{ni+\lambda} = \sum_{\mu=0}^{2\jmath+1} (-1)^{\mu} \left[\sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} p^{\kappa}_{i+(\nu-\jmath)} \right] \Gamma(t+\jmath,\mu), \qquad (3.\ 23)$$

where

$$\Gamma(t+\jmath,\mu) = \frac{\Gamma(t+\jmath+1)}{\Gamma(t+\jmath+1-\mu)\Gamma(\mu+1)}, \qquad t = \frac{2\lambda+1}{2n}.$$

Here, $\lambda = 0,1,\cdots,n-1$, $n \geq 2$ indicates the arity of the scheme (i.e binary, ternary and so on), $\jmath \geq 0$ indicates how many points we require in the scheme and $\kappa \geq 0$ indicates the subdivision level.

Remark 3.1. *Our proposed generalized form of $(2\jmath+2)$-point n-ary approximating subdivision scheme (3. 23) gives some well known approximating subdivision schemes as follows.*

- *For $\jmath = 0$ and $n = 2$ in the proposed formula (3. 23), it gives Chaikin's 2-point scheme* [4]
- *For $\jmath = 1$ and $n = 2$, (3. 23) gives Dyn's 4-point binary scheme* [9].
- *For $\jmath = 1$ and $n = 3$, (3. 23) gives Kwan's 4-point ternary scheme*[14].
- *For $\jmath = 2$ and $n = 3$, (3. 23) gives Kwan's 6-point ternary scheme*[15].

4. TENSOR PRODUCT $(2\jmath+2)$-POINT n-ARY SUBDIVISION SURFACE

Given a set of initial points $p^{\kappa}_{i,j} \in \mathbb{R}^{\aleph}$, $i,j \in \mathbb{Z}$, $\aleph \geq 2$ and $\kappa \geq 0$, the tensor product defines n-ary subdivision surface given by

$$p^{\kappa+1}_{ni+\lambda,nj+\gamma} = \sum_{r=0}^{\jmath} \sum_{s=0}^{\jmath} a_{\lambda,r} a_{\gamma,s} p^{\kappa}_{i+r,j+s}, \quad \lambda,\gamma = 0,1,\cdots,(n-1). \qquad (4.\ 24)$$

Here, $\{a_{\lambda,r}\}_{r=0}^{\jmath}$ and $\{a_{\gamma,s}\}_{s=0}^{\jmath}$ are the sets of the subdivision masks and satisfy (2. 2). Given initial points $p^{0}_{i,j}$, the tensor product (4. 24) defines a countless set of points in $\mathbb{R}^{\aleph}$ as $\kappa \to \infty$. The set of points $\{p^{\kappa}_{i,j}\}$ is related in a way with mesh points at $\left(\frac{i}{n^{\kappa}}, \frac{j}{n^{\kappa}}\right)$, $i,j \in \mathbb{Z}$. This tensor product thus defines a scheme whereby $p^{\kappa+1}_{ni+\lambda,nj+\gamma}$ are inserted at the new mesh points $\left(\frac{ni+\lambda}{n^{\kappa+1}}, \frac{nj+\gamma}{n^{\kappa+1}}\right)$ for $\lambda,\gamma = 0,1,\ldots,n-1$. Figure 1 shows both coarse and refined points of binary subdivision scheme of (i.e. n=2) (4. 24).

Now, we have the following proposed generalized form of $(2\jmath+2)$-point n-ary subdivision surface attained by tensor product using Newton interpolating polynomial.

$$p^{\kappa+1}_{n_1 i+\lambda,n_2 j+\gamma} = \sum_{\mu_1=0}^{2\jmath_1+1} \sum_{\mu_2=0}^{2\jmath_2+1} (-1)^{\mu_1+\mu_2} \left[\sum_{\nu_1=0}^{\mu_1} \sum_{\nu_2=0}^{\mu_2} (-1)^{\nu_1+\nu_2} \binom{\mu_1}{\nu_1} \binom{\mu_2}{\nu_2} p^{\kappa}_{i+(\nu_1-\jmath_1),j+(\nu_2-\jmath_2)} \right] \times \Gamma(t_1+\jmath_1,\mu_1)\Gamma(t_2+\jmath_2,\mu_2), \qquad (4.\ 25)$$

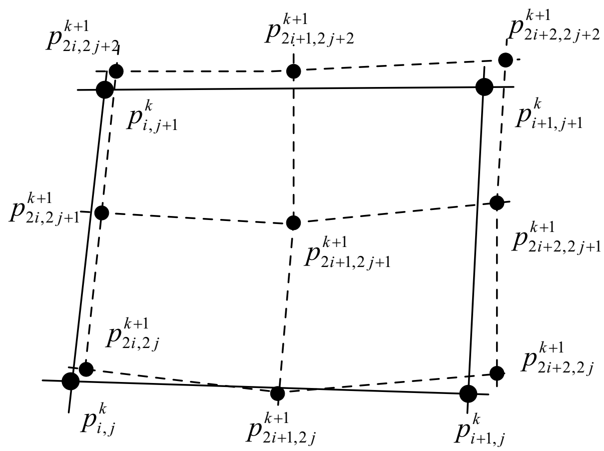


FIGURE 1. Firm lines represent the coarse polygon; spotted lines represent the refined polygon.

where

$$\Gamma(t_1+\jmath_1,\mu_1)=\frac{\Gamma(t_1+\jmath_1+1)}{\Gamma(t_1+\jmath_1+1-\mu_1)\Gamma(\mu_1+1)},$$

$$\Gamma(t_2+\jmath_2,\mu_2)=\frac{\Gamma(t_2+\jmath_2+1)}{\Gamma(t_2+\jmath_2+1-\mu_2)\Gamma(\mu_2+1)},$$

$\lambda=0,1,\cdots,n_1-1,\gamma=0,1,\cdots,n_2-1,\quad \jmath_1,\jmath_2\geq 0$, $\kappa\geq 0$ indicates the subdivision level, $t_1=\frac{2\lambda+1}{2n_1}, t_2=\frac{2\gamma+1}{2n_2}$ and $n_1,n_2=2,3,\cdots,n$.

Remark 4.1. *Our proposed generalized form of $(2\jmath+2)$-point n-ary subdivision surface attained by tensor product (4. 25) gives some well known approximating subdivision schemes as follows.*

- *By taking tensor product of Chakin's 2-point binary scheme* [4], *we get the existing Doo-Sabin scheme* [6] *which can be obtained by taking $\jmath_1,\jmath_2=0$ and $n_1,n_2=2$ in (4. 25).*
- *By setting $n_1,n_2=2$ and $\jmath_1,\jmath_2=0$ in (4. 25), we get the following mask of tensor product of 4-point binary scheme*

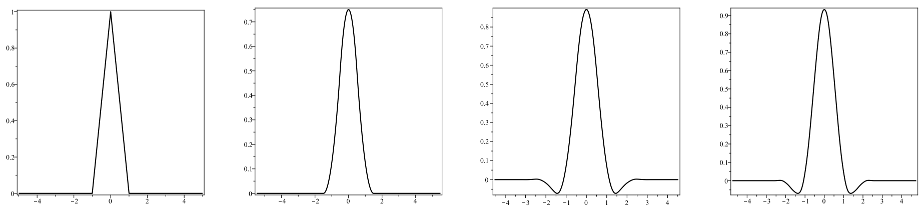
Mask = $\frac{1}{16384}\Big(25,35,35,49,-175,-525,-245,-735,-525,-175,-735,$

$-245,35,25,49,35,-175,-245,-525,-735,1225,3675,3675,11025,3675,$

$1225,11025,3675,-245,-175,-735,-525,-525,-735,-175,-245,3675,$

$11025,1225,3675,11025,3675,3675,1225,-735,-525,-245,-175,35,49,$

$25,35,-245,-735,-175,-525,-735,-245,-525,-175,49,35,35,25\Big).$



(a) (b) (c) (d)

FIGURE 2. *(a) shows the primary polygon and (b)-(d) represent fundamental limit functions of 2 and 4-point binary as well as 4-point ternary subdivision techniques respectively.*

- *Similarly by taking the values* $n_1, n_2 = 3$ *and* $\jmath_1, \jmath_2 = 1$*, in (4. 25), we get the tensor product* 4*-point ternary scheme.*

5. ANALYSIS AND APPLICATIONS

In this section, compact support and continuity of the subdivision scheme is illustrated from the following Lemmas.

Lemma 5.1. [28] *Let the points* $p^{\kappa}_{i,j}$*,* $\kappa \geq 1$ *recursively defined by* (4.24) *with given initial points* $p^0_{i,j} = p_{i,j}$*,* $i, j \in \mathbb{Z}$*, then the tensor product of the proposed scheme have four sided support regions.*

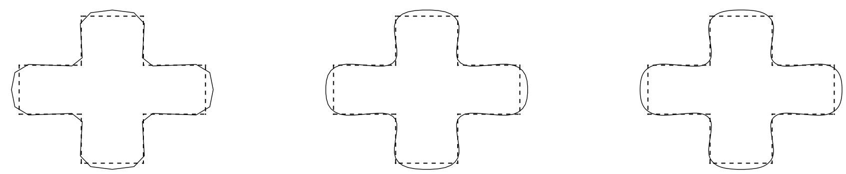
Remark 5.1. *The Doo-Sabin subdivision scheme* [6] *for surface case is obtained by taking tensor product of* [4]*. Likewise, the support of the proposed tensor product scheme (4. 25) is same as obtained by taking tensor product of the compact supports of given two regions.*

Lemma 5.2. [5] *Let the points* $p^{\kappa}_{i,j}$*,* $\kappa \geq 1$ *be recursively defined by* (4.24) *with the given initial points* $p^0_{i,j} = p_{i,j}$*,* $i, j \in \mathbb{Z}$*, then the same level of continuity is attained for tensor product schemes as their counterparts.*

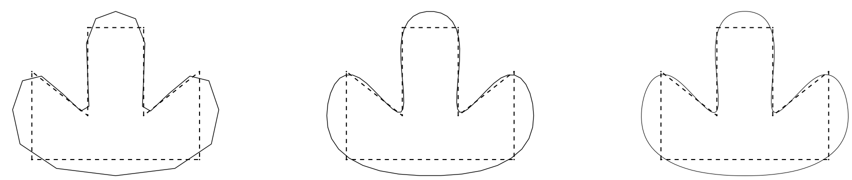
5.3. **Applications.** Figure 2 shows the basic limit functions of binary $2, 4$-point and ternary 4-point approximating subdivision schemes for curves, while visual performances of approximating subdivision schemes for curve fitting on discrete data set are shown in Figure 3 at different levels of subdivision. Figure 4 shows the basic limit functions of tensor product binary $2, 4$-point and ternary 4-point approximating subdivision schemes for surfaces, while visual performances of these schemes for surface fitting on discrete data set are shown in Figure 5 at different levels of subdivision.

6. ERROR BOUND

Subdivision process is the perfect approach for geometric and shape designing. When subdivision technique approximates the exact curve, there arises an important question just how to approximate the limit curve with its initial polygon. The method for estimating error bounds of n-ary schemes by calculating the maximal differences between

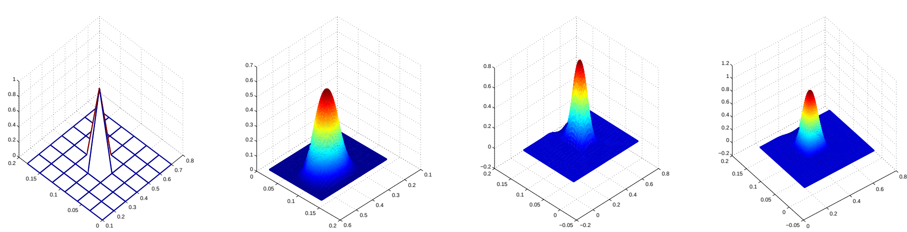


(a) 4-point 1^{st}-step (b) 2^{nd}-step (c) 3^{rd}-step



(a) 6-point 1^{st}-step (b) 2^{nd}-step (c) 3^{rd}-step

FIGURE 3. *Spotted lines show primary polygons whereas continuous curves are produced by ternary* 4*-point and* 6*-point approximating subdivision schemes.*



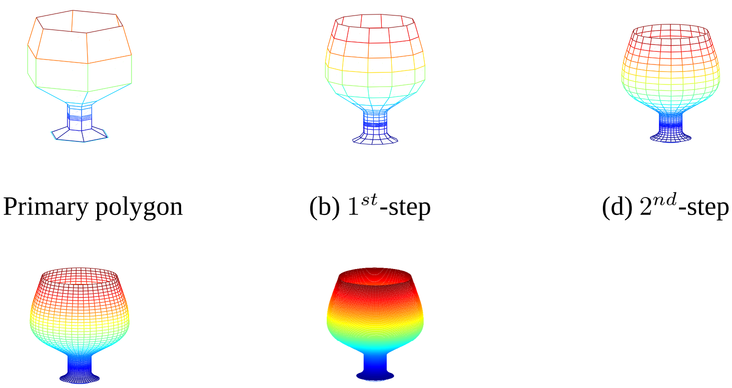
(a) (b) (c) (d)

FIGURE 4. *(a) shows the primary mesh and (b)-(d) represent basic limit functions of 2 and 4-point binary as well as 4-point ternary subdivision techniques respectively*

initial points and constants that depend on the weight of the subdivision scheme is given in [10, 22, 23]. By utilizing a similar strategy, we have the following lemma for $(2\jmath+2)$-point n-ary scheme with some expressions, inequalities and results.

Lemma 6.1. *Given initial polygon* $p_i^0 = p_i$*,* $i \in \mathbb{Z}$*, let* p_i^κ*,* $\kappa \geq 1$ *be defined by (2. 1). Suppose* p_i^κ *linearly interpolates to* p^κ *and* p^∞ *is defined to be the limit curve of (2. 1). Then the error bound after* κ*-fold subdivision between initial polygon and its limit curve is*

$$\| p^\kappa - p^\infty \|_\infty \leq \sigma\vartheta \left(\frac{(\xi)^\kappa}{1-\xi} \right), \qquad (6.\ 26)$$



(a) Primary polygon (b) 1^{st}-step (d) 2^{nd}-step

(e) 3^{rd}-step (d) Limit surface

FIGURE 5. *Performance of tensor product* 2*-point binary approximating scheme: (a), (b), (c), (d) and (e) show the initial polygon,* 1*st-,* 2*nd- and* 3*rd-subdivision level and limit surface respectively.*

where

$$\vartheta = \max_{i} \parallel p^0_{i+1} - p^0_i \parallel$$

and

$$\xi = \max_{\eta} \left\{ \left| \sum_{j=0}^{\jmath} b_{\eta,j} \right|, \eta = 0, 1, \ldots, n-1 \right\},$$

where

$$\begin{cases} b_{\eta,j} = \sum\limits_{x=0}^{j} (a_{\eta,x} - a_{\eta+1,x}), & \eta = 0, 1, \ldots, n-2, \\ b_{n-1,j} = a_{0,j} - \sum\limits_{\eta=0}^{\jmath-1} b_{\eta,j}. \end{cases}$$

Also

$$\sigma = \max_{\lambda} \left\{ \left| \sum_{j=0}^{\jmath-1} \tau_{\lambda,j} \right|, \lambda = 0, 1, \ldots, n-1 \right\},$$

where

$$\begin{cases} \tau_{\lambda,0} = \sum\limits_{x=1}^{\jmath} a_{\lambda,x} - \frac{\lambda}{n}, & \lambda = 0, 1, \ldots, n-1, \\ \tau_{\lambda,j} = \sum\limits_{x=j+1}^{\jmath} a_{\lambda,x}, & j \geq 1. \end{cases}$$

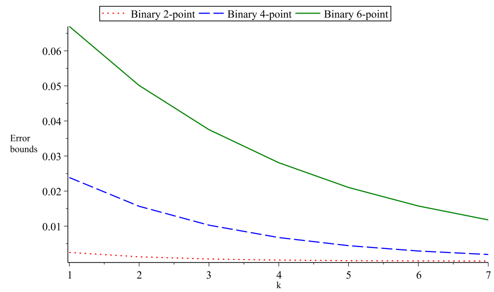


FIGURE 6. Comparison of the error bounds of 2, 4, 6-point binary approximating schemes (after κ-fold subdivision)

6.2. **Error bounds of $(2\jmath+2)$-point n-ary approximating scheme.** In this section, we have presented error bounds computed by (6. 26) between initial polygon and its limit curve after κ-fold subdivision for even points binary, ternary and quaternary schemes. From Table-1, it can be seen that error bound of 2-point scheme is less than that of 4-point scheme and error bound of 4-point scheme is less than that of 6-point scheme (for binary case) at each subdivision level. Moreover from Table-2, error bound of 2-point scheme is less than that of 4-point scheme which is less than that of 6-point scheme (for ternary case). Also in Table-3 same behavior is observed between 2,4,6-point quaternary approximating schemes as seen in Table-1 and Table-2. It is further noted that all these error bounds are computed with $\mu = 0.01$. Moreover, we have also given the graphical comparison of even points schemes for binary, ternary and quaternary approximating schemes in Figures 6, 7 and 8 respectively.

TABLE 1. *Error estimation of even-point binary schemes*

$Scheme$	$\jmath \vert \kappa$	1	2	3	4	5	6
2-point	0	0.002500	0.001250	0.000625	0.000312	0.000156	0.000078
4-point	1	0.023863	0.015660	0.010277	0.006744	0.004426	0.002904
6-point	2	0.066975	0.050133	0.037526	0.028090	0.021026	0.015739

TABLE 2. *Error estimation of even-point ternary schemes*

$Scheme$	$\jmath \vert \kappa$	1	2	3	4	5	6
2-point	0	0.001666	0.000833	0.000416	0.000208	0.000104	0.000052
4-point	1	0.009217	0.004068	0.001795	0.000792	0.000349	0.000154
6-point	2	0.022168	0.011211	0.005669	0.002867	0.001450	0.000733

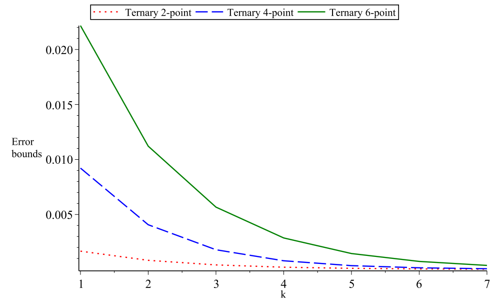


FIGURE 7. Comparison of the error bounds of 2, 4, 6-point ternary approximating schemes (after κ-fold subdivision).

TABLE 3. *Error estimation of even-point quaternary schemes*

$Scheme$	$\jmath \vert \kappa$	1	2	3	4	5	6
2-point	0	0.004167	0.001042	0.000260	0.000065	0.000016	0.000004
4-point	1	0.055921	0.018567	0.006165	0.002047	0.000679	0.000226
6-point	2	0.130828	0.049853	0.018997	0.007239	0.002758	0.001051

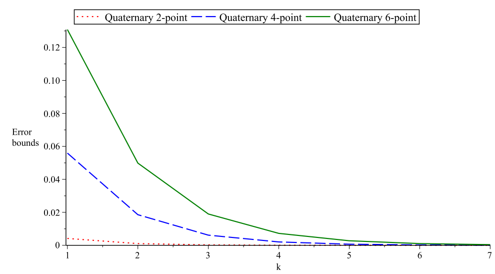


FIGURE 8. Comparison of the error bounds of 2, 4, 6-point quaternary approximating schemes (after κ-fold subdivision).

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AUTHORS CONTRIBUTION

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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