Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 51(12)(2019) pp. 53-69

Analytical Approximate Solution of Hepatitis B Epidemic Model Comparison with Vaccination

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Received: 17 July, 2019 / Accepted: 16 October, 2019 / Published online: 01 December, 2019

Abstract. In this research work, we find the analytical approximate solution of an Hepatitis B epidemic model by using Homotopy Perturbation Method (HPM). We obtaining from our solutions that Homotopy Perturbation Method (HPM) is a standout amongst the most valuable method, like just a couple of perturbation terms are sufficient for getting a reasonable accurate solution. The obtained results of Homotopy Perturbation Method (HPM) have been compare with Runge-Kutta fourth order method. Numerical justification of both methods are given.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Homotopy, Analytical, nonlinear, Runge-Kutta Method, Comparison, Numerical results..

1. Introduction

The nonlinear circumstances plays a significant role in the areas of applied mathematics, nonlinear mechanics, biochemistry, chemical engineering, life sciences and biodiversity. Since in the existence of computer programming softwares, the solution of a linear system is not a big issue. However, to finding the solution of a non linear problem persists a difficult task. The analytical methods are fastly developing, also several analytical methods and numerical have been developed [27, 15, 21, 22, 26, 18, 19, 3, 28]. But up to the present time have some scarcity and imperfection, which do not gratify mathematicians. Like nonlinear analytical technique, Homotopy Perturbation Method (HPM) is widely known method for attaining analytical inexact solutions to the differential equations. The greater part of biological circumstances in the type of mathematical models are characteristically nonlinear. Subsequently it is difficult as well as constantly difficult to locate the analytical solutions that speak to the total biological wonders. Thus, the researchers are in pursuit to discover such numerical methods or analytical technique to locate the analytical solution and rough answer for these nonlinear problems. In the numerical techniques, convergence and stability to be considered to keep away from improper outcomes or divergence. While, in the analytical perturbation technique, we have to apply the small parameter in the system. In this manner, finding the small parameter and applying it into the system are difficulties of this technique. Be that as it may, there are few restrictions with the basic perturbation technique, similar to the basic analytical method depends on the presence of a small parameter, which is hard to apply to real word problems. Along these lines, a wide range of groundbreaking mathematical techniques have been as of recently acquainted with disappear the small parameters, for example, artificial parameter technique [4, 5].

Homotopy Perturbation Method (HPM) was earliest introduced by a Chinese mathematician Ji-Huan He in 1999 [6, 7, 8, 9, 10, 11, 12, 13, 14]. Later this was applied by many authors [23, 24, 2], towards the solution of different nonlinear complications in the divergent fields of science. The potential of this method to solve linear and nonlinear problems and also this method does not need any parameter restrictions like traditional perturbation techniques. It is also germane to diverse natures of equations like Lotka-Voltera type systems, wave equation, Volterra integral equations, heat evolution equation and so forth. Much of the time the first put forward technique gives a very fast and quick convergence [25]. Therefore, Homotopy Perturbation Method (HPM) is an approximate technique which has the capacity to solve various sorts of nonlinear differential equations [1, 20].

The goal of this research is just before present the applications of the Homotopy Perturbation Method (HPM) to explain a mathematical model presented in [29]. First, we solve our problem by HPM to find the analytical approximate and as well as numerical solutions. Then, for the numerical simulation we also estimated the parameters in the model.
The article is sorted out per section. Section 2 is committed to the basic idea of HPM and the mathematical construction of the model. In Section 3 the model is solved by HPM. We presented the comparison of HPM with Runge-Kutta fourth order method and then discussed our results in section 4.

2. Analysis of HPM

To illustrate the basic idea of HPM, consider the general nonlinear differential equation.

$$A(\xi) - f(r) = 0, \quad r \in \Omega, \tag{2. 1}$$

with the boundary condition,

$$\Psi\left(\xi, \frac{\delta\xi}{\delta n}\right) = 0, \quad r \in \Gamma, \tag{2. 2}$$

where A is a general differential operator, Ψ is a boundary operator, $f(r)$ a known analytic function, Γ is the boundary of the domain Ω. The operator A is partitioned into linear part L and nonlinear part N. In this way, equation (1) can be written as,

$$L(\xi) + N(\xi) - f(r) = 0. \tag{2. 3}$$

By utilizing the homotopy technique, one can construct a homotopy $\upsilon(r,p)$:$\Omega \times [0,1] \longrightarrow R$ which satisfies

$$H(\upsilon, p) = (1-p)[L(\upsilon) - L(\xi_0)] + p[A(\upsilon) - f(r)] = 0, \tag{2. 4}$$

or

$$H(\upsilon, p) = L(\upsilon) - L(\xi_0) + pL(\xi_0) + p[N(\upsilon) - f(r)] = 0, \tag{2. 5}$$

where $p \in [0,1]$ is an embedding parameter and ξ_0 is the initial approximation of the given equation that satisfies the boundary conditions. Unmistakably, we have

$$H(\upsilon, 0) = L(\upsilon) - L(\xi_0) = 0, \tag{2. 6}$$

$$H(\upsilon, 1) = A(\upsilon) - f(r) = 0, \tag{2. 7}$$

The changing process of p from zero to one is just that of $\upsilon(r,p)$ changing from $\xi_0(r)$ to $\xi(r)$. This is called deformation and also $L(\upsilon) - L(\xi_0)$ and $A(\upsilon) - f(r)$ are called homotopic in topology. If the embedding parameter p $(0 \leq p \leq 1)$ is considered as a small parameter, applying the classical perturbation technique, we can naturally expect that the solution of the equation can be given as a power series in p,

$$\upsilon = \upsilon_0 + p\upsilon_1 + p^2\upsilon_2 + p^3\upsilon_3 + ... \tag{2. 8}$$

Setting $p = 1$ results in the approximate solution as

$$\upsilon = \lim_{p\to 1} = \upsilon_0 + \upsilon_1 + \upsilon_2 + \upsilon_3 + ... \tag{2. 9}$$

3. Mathematical Model

In this section, we consider the model presented in [29]. This model is based on the transmission of hepatitis B virus in China see the flowchart in figure 1. According to the author, he divided the total population in to six epidemiological classes: namely, the population of susceptible infection S; latently infected L; acute infection I; carriers C; recovered and with protective immunity R; and immune following vaccination individuals V. The model is given by the following system of ordinary differential equations:

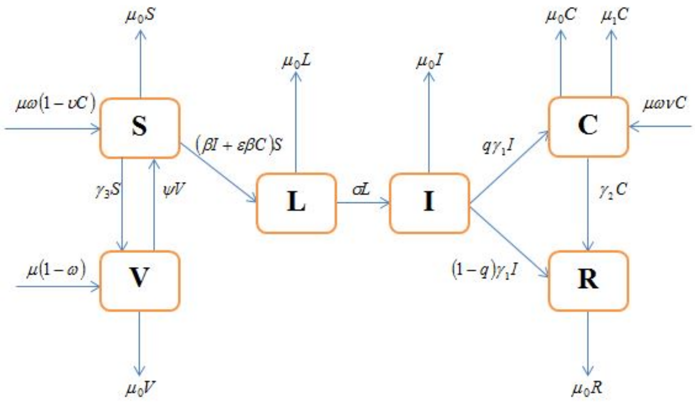

FIGURE 1. Flowchart of HBV transmission in a population

$$\begin{cases} \dfrac{dS}{dt} = \mu\omega(1-\nu C) + \psi V - (\mu_0 + \beta I + \epsilon\beta C + \gamma_3)S, \\ \dfrac{dL}{dt} = (\beta I + \epsilon\beta C)S - (\mu_0 + \sigma)L, \\ \dfrac{dI}{dt} = \sigma L - (\mu_0 + \gamma_1)I, \\ \dfrac{dC}{dt} = \mu\omega\nu C + q\gamma_1 I - (\mu_0 + \mu_1 + \gamma_2)C, \\ \dfrac{dR}{dt} = \gamma_2 C + (1-q)\gamma_1 I - \mu_0 R, \\ \dfrac{dV}{dt} = \mu(1-\omega) + \gamma_3 S - (\mu_0 + \psi)V, \end{cases} \tag{3. 10}$$

with the initial conditions

$$S(0) = S_0, \quad L(0) = L_0, \quad I(0) = I_0, \quad C(0) = C_0, \quad R(0) = R_0, \quad V(0) = V_0. \tag{3. 11}$$

From equation (3. 10), μ is the birth rate, μ_0 natural mortality rate, μ_1 hepatitis B virus related death rate, β transmission coefficient, ϵ reduced transmission rate, from latent to acute at a rate σ, from acute to carrier at a rate γ_1, from carrier to invulnerable at a rate γ_2, the moving rate γ_3 which acute goes to chronic, ω proportion of births lacking successful vaccination, q average probability an individual be unsuccessful to clear an acute infection and creates to carrier state, ψ waning rate of vaccine-induced immunity, ν proportion of prenatally infected.
Now we apply HPM to equation (3. 10), to obtain the following system:

$$\begin{cases} \frac{dS}{dt} - \frac{dS_0}{dt} = p[\mu\omega(1-\nu C) + \psi V - (\mu_0 + \beta I + \epsilon\beta C + \gamma_3)S - \frac{dS_0}{dt}], \\ \frac{dL}{dt} - \frac{dL_0}{dt} = p[(\beta I + \epsilon\beta C)S - (\mu_0 + \sigma)L - \frac{dL_0}{dt}], \\ \frac{dI}{dt} - \frac{dI_0}{dt} = p[\sigma L - (\mu_0 + \gamma_1)I - \frac{dI_0}{dt}], \\ \frac{dC}{dt} - \frac{dC_0}{dt} = p[\mu\omega\gamma C + q\gamma_1 I - (\mu_0 + \mu_1 + \gamma_2)C - \frac{dC_0}{dt}], \\ \frac{dR}{dt} - \frac{dR_0}{dt} = p[\gamma_2 C + (1-q)\gamma_1 I - \mu_0 R - \frac{dR_0}{dt}], \\ \frac{dV}{dt} - \frac{dV_0}{dt} = p[\mu(1-\omega) + \gamma_3 S - (\mu_0 + \psi)V - \frac{dV_0}{dt}]. \end{cases} \tag{3. 12}$$

According to HPM, now we consider the solution of the system (3. 12) in the following form,

$$\begin{cases} S = S_0 + pS_1 + p^2 S_2 + ...,\\ L = L_0 + pL_1 + p^2 L_2 + ...,\\ I = I_0 + pI_1 + p^2 I_2 + ...,\\ C = C_0 + pC_1 + p^2 C_2 + ...,\\ R = R_0 + pR_1 + p^2 R_2 + ...,\\ V = V_0 + pV_1 + p^2 V_2 + ... \end{cases} \tag{3. 13}$$

By considering the system (3. 13) in the system (3. 12) and equating the coefficient of corresponding powers of p, we get,

$$\begin{cases} \frac{dS_1}{dt} = \mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0 - \frac{dS_0}{dt}, \\ \frac{dL_1}{dt} = \beta I_0 S_0 + \epsilon\beta C_0 S_0 - \mu_0 L_0 - \sigma L_0 - \frac{dL_0}{dt}, \\ \frac{dI_1}{dt} = \sigma L_0 - \mu_0 I_0 - \gamma_1 I_0 - \frac{dI_0}{dt}, \\ \frac{dC_1}{dt} = \mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0 - \frac{dC_0}{dt}, \\ \frac{dR_1}{dt} = \gamma_2 C_0 + (1-q)\gamma_1 I_0 - \mu_0 R_0 - \frac{dR_0}{dt}, \\ \frac{dV_1}{dt} = \mu(1-\omega) + \gamma_3 S_0 - (\mu_0 + \psi)V_0 - \frac{dV_0}{dt}, \end{cases} \tag{3. 14}$$

and

$$\begin{cases} \frac{dS_2}{dt} = (-\mu\omega\nu - \epsilon\beta S_0)C_1 + \psi V_1 - \beta S_0 I_1 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_1, \\ \frac{dL_2}{dt} = \beta S_0 I_1 + (\beta I_0 + \epsilon\beta C_0)S_1 + \epsilon\beta S_0 C_1 - (\mu_0 + \sigma)L_1, \\ \frac{dI_2}{dt} = \sigma L_1 - (\mu_0 + \gamma_1)I_1, \\ \frac{dC_2}{dt} = (\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))C_1 + q\gamma_1 I_1, \\ \frac{dR_2}{dt} = \gamma_2 C_1 + (1-q)\gamma_1 I_1 - \mu_0 R_1, \\ \frac{dV_2}{dt} = \gamma_3 S_1 - (\mu_0 + \psi)V_1. \end{cases} \tag{3. 15}$$

Similarly

$$\begin{cases} \frac{dS_3}{dt} = (-\mu\omega\nu - \epsilon\beta S_0)C_2 + \psi V_2 - \beta S_0 I_2 - \beta S_1 I_1 - \epsilon\beta S_1 C_1 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_2, \\ \frac{dL_3}{dt} = \beta S_0 I_2 + \beta S_1 I_1 + (\beta I_0 + \epsilon\beta C_0)S_2 + \epsilon\beta S_0 C_2 + \epsilon\beta S_1 C_1 - (\mu_0 + \sigma)L_2, \\ \frac{dI_3}{dt} = \sigma L_2 - (\mu_0 + \gamma_1)I_2, \\ \frac{dC_3}{dt} = (\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))C_2 + q\gamma_1 I_2, \\ \frac{dR_3}{dt} = \gamma_2 C_2 + (1-q)\gamma_1 I_2 - \mu_0 R_2, \\ \frac{dV_3}{dt} = \gamma_3 S_2 - (\mu_0 + \psi)V_2. \end{cases} \tag{3. 16}$$

By putting $p = 1$ in the system (3. 13), we obtain the following HPM solution,

$$\begin{cases} S(t) = S_0 + S_1 + S_2 + \text{.........}, \\ L(t) = L_0 + L_1 + L_2 + \text{.........}, \\ I(t) = I_0 + I_1 + I_2 + \text{.........}, \\ C(t) = C_0 + C_1 + C_2 + \text{.........}, \\ R(t) = R_0 + R_1 + R_2 + \text{.........}, \\ V(t) = V_0 + V_1 + V_2 + \text{..........} \end{cases} \tag{3. 17}$$

Now we consider the following cases.

3.1. **Zeroth Order Problem or P^0.**

$$\begin{cases} S_0 = & S(0) = & 130, \\ L_0 = & L(0) = & 90, \\ I_0 = & I(0) = & 80, \\ C_0 = & C(0) = & 110, \\ R_0 = & R(0) = & 100, \\ V_0 = & V(t) = & 120. \end{cases} \tag{3. 18}$$

3.2. **First Order Problem or P^1.**

$$\begin{cases} S_1 = (\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t, \\ L_1 = (\beta I_0 S_0 + \epsilon\beta C_0 S_0 - \mu_0 L_0 - \sigma L_0)t, \\ I_1 = (\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t, \\ C_1 = (\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t, \\ R_1 = (\gamma_2 C_0 + (1-q)\gamma_1 I_0 - \mu_0 R_0)t, \\ V_1 = (\mu(1-\omega) + \gamma_3 S_0 - (\mu_0 + \psi)V_0)t. \end{cases} \tag{3. 19}$$

3.3. **Second Order Problem or P^2.**

$$\begin{cases} S_2 = (-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^2/2 + \psi(\mu(1-\omega) \\ \quad - \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^2/2 - \beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2 - (\mu_0 + \beta I_0 \\ \quad + \epsilon\beta C_0 + \gamma_3)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)t^2/2, \\ \\ L_2 = \beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2 + (\beta I_0 + \epsilon\beta C_0)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 \\ \quad - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^2/2 + \epsilon\beta S_0(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \\ \quad \mu_1 + \gamma_2)C_0)t^2/2 - (\mu_0 + \sigma)(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^2/2, \end{cases} \tag{3. 20}$$

$$\begin{cases} I_2 = \sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^2/2 - (\mu_0 + \gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2, \\ \\ C_2 = (\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^2/2 \\ \quad + q\gamma_1(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2, \\ \\ R_2 = \gamma_2(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^2/2 + (1-q)\gamma_1(\sigma L_0 - \mu_0 I_0 \\ \quad - \gamma_1 I_0)t^2/2 - \mu_0(\gamma_2 C_0 + (1-q)\gamma_1 I_0 - \mu_0 R_0)t^2/2, \\ \\ V_2 = \gamma_3(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^2/2 \\ \quad - (\mu_0 + \psi)(\mu(1-\omega) + \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^2/2. \end{cases} \tag{3. 21}$$

3.4. Third Order Problem or P^3.

$$
\begin{cases}
S_3 = (-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu - (\mu_0+\mu_1+\gamma_2))(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0+\mu_1+\gamma_2)C_0)t^3/6 + q\gamma_1 \\
\quad (-\mu\omega\nu - \epsilon\beta S_0)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \psi\gamma_3(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3) \\
\quad S_0)t^3/6 - \psi(\mu_0+\psi)(\mu(1-\omega) + \gamma_3 S_0 - (\mu_0+\psi)V_0)t^3/6 - \beta S_0\sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0+\sigma) \\
\quad L_0)t^3/6 + \beta S_0(\mu_0+\gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 - \beta(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 \\
\quad - \epsilon\beta C_0 S_0 - \gamma_3 S_0)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/3 - \epsilon\beta(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 \\
\quad - \epsilon\beta C_0 S_0 - \gamma_3 S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0+\mu_1+\gamma_2)C_0)t^3/3 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3) \\
\quad (-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0+\mu_1+\gamma_2)C_0)t^3/6 - \psi(\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3) \\
\quad (\mu(1-\omega) - \gamma_3 S_0 - (\mu_0+\psi)V_0)t^3/6 + \beta S_0(\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\
\quad + (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)^2(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)t^3/6,
\end{cases}
\tag{3. 22}
$$

$$
\begin{cases}
L_3 = \beta S_0\sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0+\sigma)L_0)t^3/6 - \beta S_0(\mu_0+\gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\
\quad + \beta(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/3 \\
\quad + (\beta I_0 + \epsilon\beta C_0)(-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0+\mu_1+\gamma_2)C_0)t^3/6 + \psi(\beta I_0 + \epsilon\beta C_0) \\
\quad (\mu(1-\omega) - \gamma_3 S_0 - (\mu_0+\psi)V_0)t^3/6 - \beta S_0(\beta I_0 + \epsilon\beta C_0)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\
\quad - (\beta I_0 + \epsilon\beta C_0)(\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 \\
\quad - \gamma_3 S_0)t^3/6 + \epsilon\beta S_0(\mu\omega\nu - (\mu_0+\mu_1+\gamma_2))(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0+\mu_1+\gamma_2)C_0)t^3/6 \\
\quad + \epsilon\beta S_0 q\gamma_1(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \epsilon\beta(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 \\
\quad - \gamma_3 S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0+\mu_1+\gamma_2)C_0)t^3/3 - (\mu_0+\sigma)\beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\
\quad - (\mu_0+\sigma)(\beta I_0 + \epsilon\beta C_0)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^3/6 \\
\quad - (\mu_0+\sigma)\epsilon\beta S_0(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0+\mu_1+\gamma_2)C_0)t^3/6 + (\mu_0+\sigma)^2(\beta S_0 I_0 + \epsilon\beta S_0 C_0 \\
\quad - (\mu_0+\sigma)L_0)t^3/6,
\end{cases}
\tag{3. 23}
$$

$$\begin{cases} I_3 = \sigma\beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \sigma(\beta I_0 + \epsilon\beta C_0)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 \\ \quad - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^3/6 + \sigma\epsilon\beta S_0(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 \\ \quad - \sigma(\mu_0 + \sigma)(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 - \sigma(\mu_0 + \gamma_1)(\beta S_0 I_0 + \epsilon\beta S_0 C_0) \\ \quad - (\mu_0 + \sigma)L_0)t^3/6 + (\mu_0 + \gamma_1)^2(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6, \\ C_3 = q\gamma_1\sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 - q\gamma_1(\mu_0 + \gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\ \quad + q\gamma_1(\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + (\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))^2 \\ \quad (\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6, \\ R_3 = \gamma_2(\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + q\gamma_1\gamma_2 \\ \quad (\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + (1-q)\gamma_1\sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 \\ \quad - (1-q)\gamma_1(\mu_0 + \gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 - \mu_0\gamma_2(\mu\omega\nu C_0 + q\gamma_1 I_0 \\ \quad - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + \mu_0(1-q)\gamma_1(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\ \quad + \mu_0^2(\gamma_2 C_0 + (1-q)\gamma_1 I_0 - \mu_0 R_0)t^3/6, \\ V_3 = \gamma_3(-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + \gamma_3\psi(\mu(1-\omega) \\ \quad - \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^3/6 - \gamma_3\beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 - \gamma_3(\mu_0 + \beta I_0 \\ \quad + \epsilon\beta C_0 + \gamma_3)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)t^3/6 \\ \quad - (\mu_0 + \psi)\gamma_3(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^3/6 + (\mu_0 + \psi)^2 \\ \quad (\mu(1-\omega) + \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^3/6. \end{cases} \tag{3. 24}$$

Now according to HPM we can conclude that

$$\begin{aligned} S(t) &= \lim_{p\to 1} S(t) = S_0 + pS_1 + p^2 S_2 + ... \\ L(t) &= \lim_{p\to 1} L(t) = L_0 + pL_1 + p^2 L_2 + ... \\ I(t) &= \lim_{p\to 1} I(t) = I_0 + pI_1 + p^2 L_2 + ... \\ C(t) &= \lim_{p\to 1} C(t) = C_0 + pC_1 + p^2 C_2 + ... \\ R(t) &= \lim_{p\to 1} R(t) = R_0 + pR_1 + p^2 R_2 + ... \\ V(t) &= \lim_{p\to 1} V(t) = V_0 + pV_1 + p^2 V_2 + ... \end{aligned}$$

$$\begin{aligned} S(t) \;=\; & S_0 + (\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t + (-\mu\omega\nu - \epsilon\beta S_0) \\ & (\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^2/2 + \psi(\mu(1-\omega) - \gamma_3 S_0 - (\mu_0 + \psi) \\ & V_0)t^2/2 - \beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)(\mu\omega - \mu\omega\nu C_0 \\ & + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)t^2/2 + (-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu - (\mu_0 + \\ & \mu_1 + \gamma_2))(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + q\gamma_1(-\mu\omega\nu - \epsilon\beta S_0) \\ & (\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \psi\gamma_3(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3) \\ & S_0)t^3/6 - \psi(\mu_0 + \psi)(\mu(1-\omega) + \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^3/6 - \beta S_0\sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 \\ & -(\mu_0 + \sigma)L_0)t^3/6 + \beta S_0(\mu_0 + \gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 - \beta(\mu\omega - \mu\omega\nu C_0 \\ & + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/3 - \epsilon\beta(\mu\omega \\ & -\mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 \\ & + \gamma_2)C_0)t^3/3 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)(-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 \\ & + \mu_1 + \gamma_2)C_0)t^3/6 - \psi(\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)\mu(1-\omega) - \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^3/6 \\ & (+\beta S_0(\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)^2 \\ & (\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)t^3/6 + ..., \end{aligned} \tag{3. 25}$$

$$\begin{aligned} L(t) \;=\; & L_0 + (\beta I_0 S_0 + \epsilon\beta C_0 S_0 - \mu_0 L_0 - \sigma L_0)t + \beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2 \\ & + (\beta I_0 + \epsilon\beta C_0)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^2/2 \\ & + \epsilon\beta S_0(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^2/2 - (\mu_0 + \sigma)(\beta S_0 I_0 + \epsilon\beta S_0 C_0 \\ & -(\mu_0 + \sigma)L_0)t^2/2 + \beta S_0\sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 - \beta S_0(\mu_0 + \gamma_1) \\ & (\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \beta(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 \\ & -\gamma_3 S_0)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/3 + (\beta I_0 + \epsilon\beta C_0)(-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 \\ & -(\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + \psi(\beta I_0 + \epsilon\beta C_0)(\mu(1-\omega) - \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^3/6 \\ & -\beta S_0(\beta I_0 + \epsilon\beta C_0)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 - (\beta I_0 + \epsilon\beta C_0)(\mu_0 + \beta I_0 + \epsilon\beta C_0 \\ & + \gamma_3)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)t^3/6 + \epsilon\beta S_0(\mu\omega\nu \\ & -(\mu_0 + \mu_1 + \gamma_2))(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + \epsilon\beta S_0 q\gamma_1(\sigma L_0 - \\ & \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \epsilon\beta(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0) \\ & (\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/3 - (\mu_0 + \sigma)\beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\ & -(\mu_0 + \sigma)(\beta I_0 + \epsilon\beta C_0)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^3/6 \\ & -(\mu_0 + \sigma)\epsilon\beta S_0(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + (\mu_0 + \sigma)^2(\beta S_0 I_0 \\ & + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 + ..., \end{aligned} \tag{3. 26}$$

$$\begin{aligned} I(t) \;=\; & I_0 + (\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t + \sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^2/2 \\ & -(\mu_0 + \gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2 + \sigma\beta S_0(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 \\ & + \sigma(\beta I_0 + \epsilon\beta C_0)(\mu\omega - \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^3/6 \\ & + \sigma\epsilon\beta S_0(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 - \sigma(\mu_0 + \sigma)(\beta S_0 I_0 \\ & + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 - \sigma(\mu_0 + \gamma_1)(\beta S_0 I_0 + \epsilon\beta S_0 C_0) - (\mu_0 + \sigma) \\ & L_0)t^3/6 + (\mu_0 + \gamma_1)^2(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + ..., \end{aligned} \tag{3. 27}$$

$$
\begin{aligned}
C(t) \;=\;& C_0 + (\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t + (\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2)) \\
& (\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^2/2 + q\gamma_1(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2 \\
& + q\gamma_1\sigma(\beta S_0 I_0 + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 - q\gamma_1(\mu_0 + \gamma_1)(\sigma L_0 - \mu_0 I_0 \\
& - \gamma_1 I_0)t^3/6 + q\gamma_1(\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \\
& (\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))^2(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + ...,
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
R(t) \;=\;& R_0 + (\gamma_2 C_0 + (1-q)\gamma_1 I_0 - \mu_0 R_0)t + \gamma_2(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 \\
& + \gamma_2)C_0)t^2/2 + (1-q)\gamma_1(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^2/2 - \mu_0(\gamma_2 C_0 + (1-q) \\
& \gamma_1 I_0 - \mu_0 R_0)t^2/2 + \gamma_2(\mu\omega\nu - (\mu_0 + \mu_1 + \gamma_2))(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \\
& \mu_1 + \gamma_2)C_0)t^3/6 + q\gamma_1\gamma_2(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + (1-q)\gamma_1\sigma(\beta S_0 I_0 \\
& + \epsilon\beta S_0 C_0 - (\mu_0 + \sigma)L_0)t^3/6 - (1-q)\gamma_1(\mu_0 + \gamma_1)(\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0) \\
& t^3/6 - \mu_0\gamma_2(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 + \gamma_2)C_0)t^3/6 + \mu_0(1-q)\gamma_1 \\
& (\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 + \mu_0^2(\gamma_2 C_0 + (1-q)\gamma_1 I_0 - \mu_0 R_0)t^3/6 + ...,
\end{aligned} \tag{3.29}
$$

$$
\begin{aligned}
V(t) \;=\;& V_0 + (\mu(1-\omega) + \gamma_3 S_0 - (\mu_0 + \psi)V_0)t + \gamma_3(\mu\omega - \mu\omega\nu C_0 + \psi V_0 \\
& - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^2/2 - (\mu_0 + \psi)(\mu(1-\omega) + \gamma_3 S_0 - \\
& (\mu_0 + \psi)V_0)t^2/2 + \gamma_3(-\mu\omega\nu - \epsilon\beta S_0)(\mu\omega\nu C_0 + q\gamma_1 I_0 - (\mu_0 + \mu_1 \\
& + \gamma_2)C_0)t^3/6 + \gamma_3\psi(\mu(1-\omega) - \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^3/6 - \gamma_3\beta S_0 \\
& (\sigma L_0 - \mu_0 I_0 - \gamma_1 I_0)t^3/6 - \gamma_3(\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)(\mu\omega - \mu\omega\nu C_0 \\
& + \psi V_0 - \mu_0 S_0 - \beta I_0 S_0 - \epsilon\beta C_0 S_0 - \gamma_3 S_0)t^3/6 - (\mu_0 + \psi)\gamma_3(\mu\omega - \\
& \mu\omega\nu C_0 + \psi V_0 - (\mu_0 + \beta I_0 + \epsilon\beta C_0 + \gamma_3)S_0)t^3/6 + (\mu_0 + \psi)^2(\mu(1-\omega) \\
& + \gamma_3 S_0 - (\mu_0 + \psi)V_0)t^3/6 + ...
\end{aligned} \tag{3. 30}
$$

4. NUMERICAL RESULTS

In this section, we solve the hepatitis B virus epidemic model numerically by using Runge-Kutta fourth order method for the numerical solution. For numerical scheme we utilize parameter values given in Table (1). We also compare the numerical and semi analytical results. So first we present a brief discussion of RK4 method in the following subsection. The comparison results are presented in Figure 2, 3, 4, 5, 6 and 7 show the population of susceptible people, latently infected people, acute infected people, carriers, population recovered with protective immunity and population immune following vaccination respectively.

4.1. **Runge-Kutta Forth Order Method.** The essential numerical method that is used in this article is Runge-kutta(RK4). The detail of Rung-Kutta (RK4) method for the solution of transmission dynamic and vaccination of Hepatitis B epidemic model is as follows; this numerical results are presented in Figure 2-7.
Suppose an initial value problem, in which we would like to approximating an unspecified function y of time t as,

$$\frac{dy}{dt} = f(t, y), \; y(t_0) = y_0.$$

TABLE 1. Interpretation of parameters and its values.

Notation	Interpretation of Parameter	Value
μ	birth rate	0.0121
μ_0	natural death rate	0.00693
μ_1	HBV related death rate	0.002
β	transmission coefficient	0.95
ϵ	transmission rate	0.16
ν	proportion of prenatally infected	0.11
σ	moving rate from latent to acute	6
γ_1	moving rate from acute to carrier	4
γ_2	moving rate from carrier to immune	0.025
γ_3	moving rate which acute goes to chronic	0.5
ω	proportion of births without successful vaccination	0.50
q	average probability an individual fails to clear an acute infection and develop to carrier state	0.885
ψ	waning rate of vaccine induced immunity	0.1

The function f and the data t_0, y_0 are given. for a small step size $h > 0$, we have

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ t_{i+1} &= t_i + h, \end{aligned} \tag{4. 31}$$

for $i = 0, 1, 2, 3, ...$, using

$$\begin{aligned} k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1), \\ k_3 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2), \\ k_4 &= f(t_i + h, y_i + hk_3). \end{aligned}$$

Here y_{i+1} is $RK4$ estimation of $y(t_{i+1})$, and the next value (y_{i+1}) is determined by the present value (y_i) plus the weighted average of four increments, where each increment is the product of the size of the interval h, and an estimated slope determined by function f on the right-hand side of the differential equation.

4.1.1 Algorithm

The algorithm of Runge-Kutta fourth order method for the system (3. 10) is given as:
$S_0(t) = 130$, $L_0(t) = 90$, $I_0(t) = 80$, $C_0(t) = 110$, $R_0(t) = 100$, $V_0(t) = 120$;

for $i = 0, 1, 2, 3, ...,$

$$\begin{aligned} S_{i+1} &= S_i + \frac{h}{6}(g_1 + 2g_2 + 2g_3 + g_4), \\ L_{i+1} &= L_i + \frac{h}{6}(h_1 + 2h_2 + 2h_3 + h_4), \\ I_{i+1} &= I_i + \frac{h}{6}(j_1 + j_2 + 2j_3 + j_4), \\ C_{i+1} &= C_i + \frac{h}{6}(q_1 + 2q_2 + 2q_3 + q_4), \\ R_{i+1} &= R_i + \frac{h}{6}(u_1 + 2u_2 + 2u_3 + u_4), \\ V_{i+1} &= V_i + \frac{h}{6}(w_1 + 2w_2 + 2w_3 + w_4). \end{aligned} \quad (4.\ 32)$$

4.2. **Comparison Analysis.** In this section, we compare HPM solution with the numerical solution obtained by RK4 method. The results in detail are given in the following tables and figures. Also an error analysis is carried out for the HPM solution and RK4 solution. Figures 2-7 and tables 2-7, presents a comparison between analytical results obtained by HPM and numerical solutions achieved from forth order Runge Kutta method. According to Figures 2-7 and tables 2-7, the obtained results have good agreement with numerical solutions. From the tables and figures it is clear that, our solution obtained by HPM is compatible with the numerical solution obtained by RK4 method. An error analysis is carried out for HPM solution and numerical solution.

TABLE 2. Susceptible S(t).

Comparison Table			
Time (t)	$X(t)_{RK4}$	$X(t)_{HPM}$	Error
0.00000	130.00000	130.00000	0.00000
0.00200	107.84536	107.83774	0.00762
0.00400	089.33916	089.22366	0.11550
0.00600	073.87612	073.32135	0.55477
0.00800	060.96234	059.29443	1.66791
0.01000	050.18993	046.30650	3.88343

TABLE 3. Latent L(t).

Comparison Table			
Time (t)	$Y(t)_{RK4}$	$Y(t)_{HPM}$	Error
0.00000	090.00000	090.00000	0.00000
0.00200	110.84757	110.85555	0.00798
0.00400	127.84069	127.96208	0.12139
0.00600	141.62354	142.20723	0.58369
0.00800	152.72251	154.47864	1.75613
0.01000	161.57265	165.66395	4.09130

TABLE 4. Acute I(t).

Comparison Table			
Time (t)	$Z(t)_{RK4}$	$Z(t)_{HPM}$	Error
0.00000	80.00000	80.00000	0.00000
0.00200	80.56609	80.56564	0.00045
0.00400	81.35306	81.34619	0.00687
0.00600	82.31706	82.28378	0.03328
0.00800	83.42157	83.32058	0.10099
0.01000	84.63607	84.39851	0.23756

TABLE 5. Carriers C(t).

Comparison Table			
Time (t)	$R(t)_{RK4}$	$R(t)_{HPM}$	Error
0.00000	110.00000	110.00000	0.00000
0.00200	110.56092	110.55939	0.00153
0.00400	111.12662	111.12069	0.00593
0.00600	111.69851	111.68583	0.01268
0.00800	112.27770	112.25674	0.02096
0.01000	112.86508	112.83535	0.02973

TABLE 6. Recovered R(t).

Comparison Table			
Time (t)	$R(t)_{RK4}$	$R(t)_{HPM}$	Error
0.00000	100.00000	100.00000	0.00000
0.00200	100.07796	100.07797	0.00001
0.00400	100.15659	100.15662	0.00003
0.00600	100.23604	100.23621	0.00017
0.00800	100.31648	100.31697	0.00049
0.01000	100.39802	100.39918	0.00116

TABLE 7. Vaccinated V(t).

Comparison Table			
Time (t)	$R(t)_{RK4}$	$R(t)_{HPM}$	Error
0.00000	120.00000	120.00000	0.00000
0.00200	120.09292	120.09296	0.00004
0.00400	120.16556	120.16607	0.00051
0.00600	120.22124	120.22373	0.00249
0.00800	120.26277	120.27031	0.00754
0.01000	120.29247	120.31022	0.01775

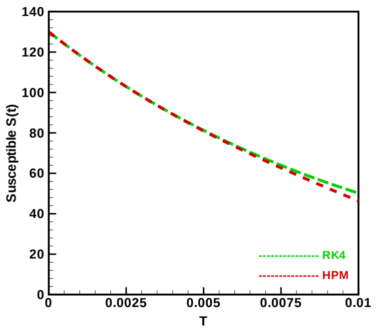

FIGURE 2. The plot represents the solution curve of S(t) by RK4 and HPM.

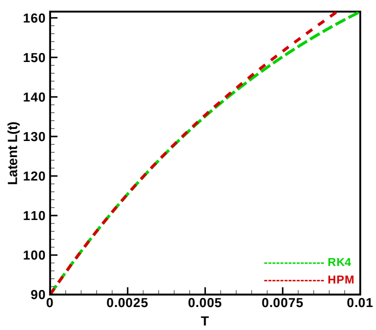

FIGURE 3. The plot represents the solution curve of L(t) by RK4 and HPM.

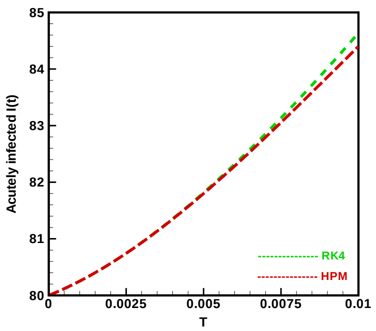

FIGURE 4. The plot represents the solution curve of I(t) by RK4 and HPM.

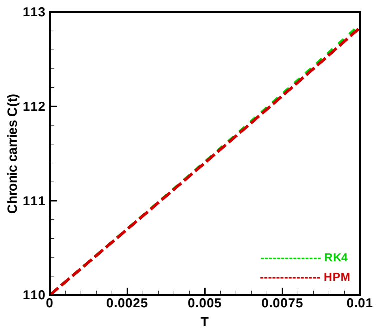

FIGURE 5. The plot represents the solution curve of C(t) by RK4 and HPM.

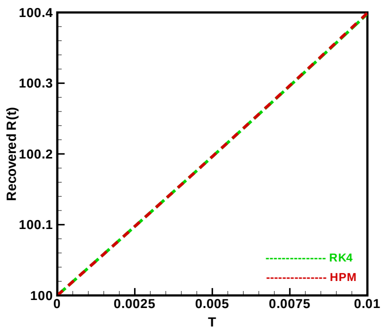

FIGURE 6. The plot represents the solution curve of R(t) by RK4 and HPM.

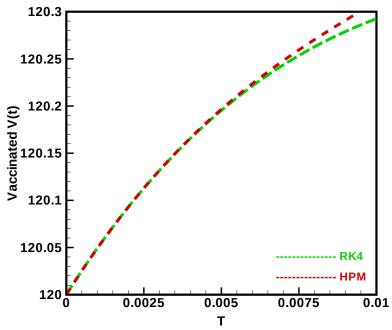

FIGURE 7. The plot represents the solution curve of V(t) by RK4 and HPM.

5. CONCLUSION

In this article, the solution of hepatitis B epidemic model is accomplished. We used a semi analytical approach for the solution of proposed model, that is HPM. Once we obtained the solution by Homotopy Perturbation Method (HPM), the comparison with Runge-Kutta method were given. A good agreement has been seen between Runge-Kutta method of order four and the semi-analytic HPM. Moreover, HPM is working without using linearization, discretization, or restrictive assumptions. In addition, this semi-analytical technique does not require too many orders solution just a few perturbation terms are sufficient for reasonable accurate solution. Which shows that HPM is a powerful technique for the solution of such type of nonlinear epidemic models.

In future work we will consider the convergence analysis of the solutions of epidemic problems using the Homotopy Perturbation Method (HPM) and RK4 methods. Work on such issues will be reported in a near future publication.

6. ACKNOWLEDGMENT

This work has been partially supported by Higher education Commission (HEC) of Pakistan under the project No.20-1983/R and D/HEC/11.

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