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Action of the möbius group $M = \langle x, y : x^2 = y^6 = 1 \rangle$ on certain real quadratic fields

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Abstract. Let $C' = C \cup \{\infty\}$ be the extended complex plane and $M = \langle x, y : x^2 = y^6 = 1 \rangle$, where $x(z) = \frac{-1}{3z}$ and $y(z) = \frac{-1}{3(z+1)}$ are the linear fractional transformations from $C' \to C'$. Let m be a square-free positive integer. Then $Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \neq 0, b = \frac{a^2-n}{c} \in Z$ and $(a, b, c) = 1\}$ where $n = k^2m$, is a proper subset of $Q(\sqrt{m})$ for all $k \in N$. For non-square $n = 3^h \prod_{i=1}^{r} p_i^{k_i}$, it was proved in an earlier paper by the same authors that the set $Q^{'''}(\sqrt{n}) = \{\frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 3\}$ is M-set $\forall\ h \geq 0$ whereas if $h = 0$ or 1, then $Q^{***}\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})\ and\ 3 \mid c\}$ is an M-subset of $Q^{'''}(\sqrt{n}) = Q^*(\sqrt{n}) \cup Q^{***}(\sqrt{9n})$. In this paper we prove that if $h \geq 2$, then $Q^{'''}(\sqrt{n}) = (Q^*(\sqrt{\frac{n}{9}}) \backslash Q^{***}(\sqrt{\frac{n}{9}})) \cup Q^*(\sqrt{n}) \cup Q^{***}(\sqrt{9n})$ and also determine its proper M-subsets. In particular $Q(\sqrt{m}) \setminus Q = \cup Q^{'''}(\sqrt{k^2m})$ for all $k \in N$.

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1. Introduction

Throughout the paper we take m as a square free positive integer. Since every element of $Q(\sqrt{m}) \setminus Q$ can be expressed uniquely as $\frac{a+\sqrt{n}}{c}$, where $n = k^2m$, k is any positive integer and $a, b = \frac{a^2-n}{c}$ and c are relatively prime integers and we denote it by $\alpha_n(a, b, c)$ or $\alpha(a, b, c)$. Then

$$Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c, b = \frac{a^2-n}{c} \in Z \text{ and } (a, b, c) = 1\},$$

$$Q^{'''}(\sqrt{n}) = \{\frac{\alpha}{t} : \alpha \in Q^*(\sqrt{n}), t = 1, 3\},$$

$$Q^{***}(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \text{ and } 3 \mid c\}$$

are subsets of the real quadratic field $Q(\sqrt{m})$ for all n and $Q(\sqrt{m}) \setminus Q$ is a disjoint union of $Q^*(\sqrt{n})$ for all n. If $\alpha(a, b, c) \in Q^*(\sqrt{n})$ and its conjugate $\overline{\alpha}$ have opposite signs then α is called an ambiguous number [7]. A non-empty set Ω with an action of a group G on it, is said to be a G -set. We say that Ω is a transitive G-set if, for any p, q in Ω there exists a g in G such that $p^g = q$.

We are interested in linear-fractional transformations x, y satisfying the relations $x^2 = y^r = 1$, with a view to studying an action of the group $\langle x, y\rangle$ on real quadratic fields. If $y : z \to \frac{az+b}{cz+d}$ is to act on all real quadratic fields then a, b, c, d must be rational numbers, and can be taken to be integers. Thus $\frac{(a+b)^2}{ad-bc}$ is rational. But if $z \to \frac{az+b}{cz+d}$ is of order of r, one must have $\frac{(a+b)^2}{ad-bc} = \omega + \omega^{-1} + 2$, where ω is a primitive r-th root of unity. Now $\omega + \omega^{-1}$ is rational, for a primitive r-th root, only if $r = 1, 2, 3, 4$ or 6, so that these are the only possible orders of y. The group $\langle x, y : x^2 = y^r = 1\rangle$ is cyclic of order 2 or D_∞(an infinite dihedral group) according as $r = 1$ or 2. For $r = 3$, the group $\langle x, y\rangle$ is the modular group $PSL(2, Z)$. The fractional linear transformations x, y with $x(z) = \frac{-1}{3z}$ and $y(z) = \frac{-1}{3(z+1)}$ generate a subgroup M of the modular group which is isomorphic to the abstract group $\langle x, y : x^2 = y^6 = 1\rangle$. It is a standard example from the theory of the modular group. It has been shown in [10] that the action of M on the rational projective line $Q \cup \{\infty\}$ is transitive.

In our case the set $Q(\sqrt{m}) \setminus Q$ is an M-set. It is noted that M is the free product of $C_2 = \langle x : x^2 = 1\rangle$ and $C_6 = \langle x : y^6 = 1\rangle$. The action of the modular group $PSL(2, Z)$ on the real quadratic fields has been discussed in detail in [1, 6, 8, 9, 11, 12]. The actual number of ambiguous numbers in $Q^*(\sqrt{n})$ has been discussed in [8] as a function of n.

In a recent paper [11], the authors have investigated that the cardinality of the set E_p, p a prime factor of n, consisting of all classes $[a, b, c](mod\, p)$ of the elements of $Q^*(\sqrt{n})$ is $p^3 - 1$ and obtained two proper G-subsets of $Q^*(\sqrt{n})$ corresponding to each odd prime divisor of n. The same authors in [12] have determined the cardinality of the set E_{p^r}, $r \geq 1$, consisting of all classes $[a, b, c](mod\, p^r)$ of the elements of $Q^*(\sqrt{n})$ and have determined, for each non-square n, the G-subsets of an invariant subset $Q^*(\sqrt{n})$ of $Q(\sqrt{m}) \setminus Q$ under the modular group action by using classes $[a, b, c](mod\ n)$. Real quadratic irrational numbers under the action of the group M have been studied in [3, 4, 5, 7, 10]. Closed paths in the coset diagrams under the action of a proper subgroup of M on $Q(\sqrt{m})$ have been discussed in [4]. M. Aslam Malik *et al.* in [2] have studied the action of $H = \langle x, y : x^2 = y^4 = 1\rangle$, where $x(z) = \frac{-1}{2z}$ and $y(z) = \frac{-1}{2(z+1)}$, on $Q(\sqrt{m}) \setminus Q$. The same authors, in [3], have discussed the properties of real quadratic irrational numbers under the action of the group M. The authors proved, in [3], that if $n \equiv 1, 3, 4, 6$ or $7(mod\, 9)$

then $Q^{***}(\sqrt{n})$ is an M-subset of $Q(\sqrt{m}) \setminus Q$ and $Q^{'''}(\sqrt{n}) = Q^{*}(\sqrt{n}) \cup Q^{***}(\sqrt{9n})$.

In this paper we extend these results for all non-square integers n and give some modifications of Lemma 1.1 of [3] for the case $n \equiv 0(mod\ 9)$ and prove that $Q^{'''}(\sqrt{n}) = (Q^{*}(\sqrt{\frac{n}{9}}) \setminus Q^{***}(\sqrt{\frac{n}{9}})) \cup Q^{*}(\sqrt{n}) \cup Q^{***}(\sqrt{9n})$ which shows that $Q(\sqrt{m}) \setminus Q$ is the union of $Q^{'''}(\sqrt{k^2m})\ \forall k \in N$. However if n and n' are two distinct non-square positive integers then $Q^{*}(\sqrt{n}) \cap Q^{*}(\sqrt{n'}) = \phi$ whereas $Q^{'''}(\sqrt{n}) \cap Q^{'''}(\sqrt{n'})$ may or may not be empty. In particular $Q^{'''}(\sqrt{n}) \cap Q^{'''}(\sqrt{9n})$ is not empty. In fact we prove that a superset namely

$$Q^{***}(\sqrt{9n}) \cup \{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^{*}(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\}$$

of $Q^{***}(\sqrt{9n})$ is an M-subset of $Q(\sqrt{m}) \setminus Q$.

We have also found M-subsets of $Q^{'''}(\sqrt{n})$ such that these may or may not be transitive. However they help in determining the transitive M-subsets (M-orbits). The notation is standard and we follow [3], [9], [11] and [12]. In particular $(\cdot/\cdot)$ denotes the Legendre symbol and $x(Y) = \{\frac{-1}{3\alpha} : \alpha \in Y\}$ for each subset Y of $Q(\sqrt{m}) \setminus Q$. Throughout this paper, n denotes a non-square positive integer and α denotes $\frac{a+\sqrt{n}}{c}$ with $b = \frac{a^2-n}{c}$ such that $(a, b, c) = 1$.

2. Preliminaries

The following results of [3], [11] and [12] will be used in the sequel.

Lemma 1. *([3]). Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^{*}(\sqrt{n})$ with $b = \frac{a^2-n}{c}$. Then:*
1. If $n \not\equiv 0(mod\ 9)$ then $\frac{\alpha}{3} \in Q^{***}(\sqrt{n})$ if and only if $3 \mid b$.
2. $\frac{\alpha}{3} \in Q^{***}(\sqrt{9n})$ if and only if $3 \nmid b$.

Theorem 2. *([3]) The set $Q^{'''}(\sqrt{n}) = \{\frac{\alpha}{t} : \alpha \in Q^{*}(\sqrt{n}),\ t = 1, 3\}$ is invariant under the action of M.*

Theorem 3. *(see [3]) For each $n \equiv 1, 3, 4, 6$ or $7(mod\ 9)$,*

$$Q^{***}\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : \frac{a+\sqrt{n}}{c} \in Q^{*}(\sqrt{n}) \text{ and } 3 \mid c\}$$

is an M-subset of $Q^{'''}(\sqrt{n})$.

Corollary 4. *([3]) $Q^{***}(\sqrt{n}) = \emptyset$ if and only if $n \equiv 2(mod\ 3)$.*

It is well known that $G = \langle x, y : x^2 = y^3 = 1\rangle$ represents the modular group, where $x(z) = \frac{-1}{z}, y(z) = \frac{z-1}{z}$ are linear fractional transformations.

Theorem 5. *([11]) Let p be an odd prime factor of n. Then both of $S_1^p = \{\alpha \in Q^{*}(\sqrt{n}) : (b/p)$ or $(c/p) = 1\}$ and $S_2^p = \{\alpha \in Q^{*}(\sqrt{n}) : (b/p)$ or $(c/p) = -1\}$ are G-subsets of $Q^{*}(\sqrt{n})$. In particular, these are the only G-subsets of $Q^{*}(\sqrt{n})$ depending upon classes $[a, b, c]$ modulo p.*

Theorem 6. *([12]) Let $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ where $p_1, p_2, ..., p_r$ are distinct odd primes such that n is not equal to a single prime congruent to 1 modulo 8. Then the number of G-subsets of $Q^{*}(\sqrt{n})$ is 2^r namely $S_{1\leq i_1,i_2,i_3,...,i_r\leq 2}$ if $k = 0$ or 1. Moreover if $k \geq 2$, then each G-subset X of these G-subsets further splits into two proper G-subsets $\{\alpha \in X : b$ or $c \equiv 1(mod\ 4)\}$ and $\{\alpha \in X : b$ or $c \equiv -1(mod\ 4)\}$. Thus the number of G-subsets of $Q^{*}(\sqrt{n})$ is 2^{r+1} if $k \geq 2$. More precisely these are the only G-subsets of $Q^{*}(\sqrt{n})$ depending upon classes $[a, b, c]$ modulo n.*

3. Action of $M = \langle x, y : x^2 = y^6 = 1\rangle$ on $Q^{'''}(\sqrt{n})$

In this section we establish that if n contains r distinct prime factors then $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ is the disjoint union of 2^r subsets which are invariant under the action of M. However these M invariant subsets may further split into transitive M-subsets (M-orbits) of $Q^{'''}(\sqrt{n})$, for example $Q^{'''}(\sqrt{37})$ splits into twelve orbits namely $(\sqrt{37})^M$, $(-\sqrt{37})^M$, $(\frac{1+\sqrt{37}}{4})^M$, $(\frac{-1+\sqrt{37}}{4})^M$, $(\frac{1+\sqrt{37}}{-4})^M$, $(\frac{-1+\sqrt{37}}{-4})^M$, $(\frac{1+\sqrt{37}}{3})^M$, $(\frac{-1+\sqrt{37}}{-3})^M$, $(\frac{1+\sqrt{37}}{6})^M$, $(\frac{-1+\sqrt{37}}{-6})^M$, $(\frac{1+\sqrt{37}}{2})^M$ and $(\frac{1+\sqrt{37}}{-2})^M$. The first six orbits are contained in $A_1^{37} \cup x(A_1^{37})$ and last four orbits are contained in $A_2^{37} \cup x(A_2^{37})$ where $A_1^{37} = S_1^{37} \setminus Q^{***}(\sqrt{37})$ and $A_2^{37} = S_2^{37} \setminus Q^{***}(\sqrt{37})$.

Lemma 7. *Let $n \equiv 1, 3, 4, 6$ or $7(mod\ 9)$. Let $Y = S \setminus Q^{***}(\sqrt{n})$ where S is any G-subset of $Q^{*}(\sqrt{n})$. Then $Y \cup\ x(Y)$ is an M-subset of $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$.*

Proof. : By Theorem 3, we know that $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ is an M-set. For any $\alpha \in Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$, Lemma 7 follows from the equations $x(\alpha) = \frac{-1}{3\alpha}$, $x(\frac{-1}{3\alpha}) = \alpha$, $y(\alpha) = \frac{-1}{3(\alpha+1)} = \frac{-1}{3\alpha'}$, where $\alpha' = \alpha + 1$ and $y(\frac{-1}{3\alpha}) = \frac{-1}{3\beta}$, where $\beta = \frac{-1}{3\alpha} + 1$. Since every element of the group $M = \langle x, y : x^2 = y^6 = 1\rangle$ is a word in the generators x, y of the group M and the transformations $\alpha \longmapsto \alpha + 1$, $\alpha \longmapsto \alpha - 1$ belong to both of the groups G and M. □

The following corollary is an immediate consequence of Lemma 7 since we know by Corollary 4 that $Q^{***}(\sqrt{n}) = \emptyset$ if and only if $n \equiv 2(mod\ 3)$.

Corollary 8. *Let $n \equiv 2(mod\ 3)$. Let S be any G-subset of $Q^{*}(\sqrt{n})$. Then $S \cup x(S)$ is an M-subset of $Q^{'''}(\sqrt{n})$.*

Theorem 9. *Let $n \equiv 1, 3, 4, 6$ or $7(mod\ 9)$ be a non-square positive integer such that $p \mid n$. Let $A_1^p = S_1^p \setminus Q^{***}(\sqrt{n})$ and $A_2^p = S_2^p \setminus Q^{***}(\sqrt{n})$. Then both of $A_1^p \cup x(A_1^p)$ and $A_2^p \cup x(A_2^p)$ are M-subsets of $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$. Consequently the action of M on $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ is intransitive.*

Proof. : follows from Theorem 5 and Lemma 7. □

We now extend Theorem 9 for each non-square n.

Theorem 10. *Let $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, ..., p_r$ are distinct odd primes and $k = 0$ or 1. Let $A_{1 \leq i_1, i_2, i_3, ..., i_r \leq 2} = S_{1 \leq i_1, i_2, i_3, ..., i_r \leq 2} \setminus Q^{***}(\sqrt{n})$. Then $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ is the disjoint union of 2^r subsets $A_{1 \leq i_1, i_2, i_3, ..., i_r \leq 2} \cup x(A_{1 \leq i_1, i_2, i_3, ..., i_r \leq 2})$ which are invariant under the action of M. More precisely these are the only M-subsets of $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ depending upon classes $[a, b, c]$ modulo n.*

Proof. : follows directly from Theorem 6 and Lemma 7. □

Theorem 11. *Let $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, ..., p_r$ are distinct odd primes and $k \geq 2$. If S is any of the G-subsets given in Theorem 6. Let $A = S \setminus Q^{***}(\sqrt{n})$. Then $A \cup x(A)$ is M-subset of $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$. More precisely these are the only M-subsets of $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ depending upon classes $[a, b, c]$ modulo n.*

Proof. : Proof follows from Theorem 6 and Lemma 7. □

If $n \equiv 0(mod\ 3)$, then by Theorem 5, $S = \{\alpha \in X : c$ or $b \equiv 1(mod\ 3)\}$ and $-S = \{\alpha \in X : c$ or $b \equiv -1(mod\ 3)\}$ are G-subsets whereas if $n \not\equiv 0(mod\ 3)$, then S

and $-S$ are not G-subsets of $Q^*(\sqrt{n})$. However the following lemma shows that $S \cup x(S)$ and $-S \cup x(-S)$ are distinct M-subsets of $Q^{'''}(\sqrt{n})$.

Lemma 12. *If $n \not\equiv 0(mod\ 9)$ and Y be any of the G-subsets of $Q^*(\sqrt{n})$. Let $X = Y \setminus Q^{***}(\sqrt{n})$. Let $S = \{\alpha \in X : c \text{ or } b \equiv 1(mod\ 3)\}$ and $-S = \{\alpha \in X : c \text{ or } b \equiv -1(mod\ 3)\}$. Then $S \cup x(S)$ and $-S \cup x(-S)$ are both disjoint M-subsets of $X \cup x(X)$. Consequently the action of M on each of $X \cup x(X)$ is intransitive.*

If $n \equiv 2, 5$ or $8(mod\ 9)$ then, by Corollary 4, $Q^{***}(\sqrt{n})$ is empty. But if $n \equiv 1, 3, 4, 6$ or $7(mod\ 9)$, then, by Theorem 3, $Q^{***}(\sqrt{n})$ is an M-subset of $Q^{'''}(\sqrt{n})$. If $n \equiv 0(mod\ 9)$, then $Q^{***}(\sqrt{n})$ is not an M-subset of $Q^{'''}(\sqrt{n})$. Instead we later prove that $Q^{***}(\sqrt{9n}) \cup \{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\}$ is an M-subset of $Q^{'''}(\sqrt{n})$. For this we need to establish the following results.

Lemma 13. *Let $n \equiv 1, 3, 4, 6$ or $7(mod\ 9)$. Then*
1. $Q^{***}(\sqrt{9n}) = Q^{'''}(\sqrt{n}) \setminus Q^*(\sqrt{n})$ and
2. $Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n}) = \{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\}$.

Proof. : 1. Let $\frac{a+\sqrt{9n}}{c} \in Q^{***}(\sqrt{9n}) = \{\frac{a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n})$ and $3 \mid c\}$. Then $\frac{a^2-9n}{c}$ and $\frac{c}{3}$ are both integers and $(a, \frac{a^2-9n}{c}, c) = 1$. As c and $9n$ are both divisible by 3, so $3 \mid a$. Let $a = 3a', c = 3c'$. Now $\frac{a^2-9n}{c} = 3(\frac{a'^2-n}{c'})$ is not divisible by 3 because otherwise $(a, \frac{a^2-9n}{c}, c) \neq 1$. So $c' = 3c''$. This shows that $\frac{(a')^2-n}{c''}$ is an integer, while $\frac{(a')^2-n}{c'}$ is not an integer for otherwise $\frac{a^2-9n}{c}$ is divisible by 3, a contradiction. Also $(a, \frac{a^2-9n}{c}, c) = 1 \Leftrightarrow (a', \frac{(a')^2-n}{c''}, c'') = 1$. Therefore $\frac{a+\sqrt{9n}}{c} = \frac{a'+\sqrt{n}}{c'} = \frac{a'+\sqrt{n}}{3c''}$, where $\frac{a'+\sqrt{n}}{c''}$ belongs to $Q^*(\sqrt{n})$. Thus $\frac{a+\sqrt{9n}}{c}$ belongs to $Q^{'''}(\sqrt{n}) \setminus Q^*(\sqrt{n})$.

Conversely let $\frac{a+\sqrt{n}}{3c} \in Q^{'''}(\sqrt{n}) \setminus Q^*(\sqrt{n})$. Then , by Lemma 1, $\frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ such that $\frac{a^2-n}{c}$ is not divisible by 3 and hence $\frac{a+\sqrt{n}}{3c} = \frac{3a+\sqrt{9n}}{9c}$ belongs to $Q^*(\sqrt{9n})$. Obviously $\frac{a+\sqrt{n}}{3c}$ belongs to $Q^{***}(\sqrt{9n})$. This completes the first part of Lemma 13.
2. We now prove that $\{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\} = Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$. For this let $\frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})$. Then $\frac{9a^2-9n}{c}$ is an integer and $(3a, \frac{9a^2-9n}{c}, c) = 1$. As $3 \nmid c$ so $\frac{9a^2-9n}{c} = 9(\frac{a^2-n}{c})$ is an integer if and only if $(\frac{a^2-n}{c})$ is an integer and also $(3a, \frac{9a^2-9n}{c}, c) = 1 \Leftrightarrow (a, \frac{a^2-n}{c}, c) = 1$. This implies that $\frac{3a+\sqrt{9n}}{3c} = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$. Conversely suppose that $\frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$. Then clearly c is not divisible by 3 and $(a, \frac{a^2-n}{c}, c) = 1$. Also $(a, \frac{a^2-n}{c}, c) = 1 \Leftrightarrow (3a, \frac{9a^2-9n}{c}, c) = 1$. Thus $\frac{a+\sqrt{n}}{c} = \frac{3a+\sqrt{9n}}{3c} = \frac{1}{3}(\frac{3a+\sqrt{9n}}{c})$, where $\frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})$. Hence the result.

The following theorem is an extension of Lemma 13 to the case for all non-square positive integers $n \equiv 0(mod\ 9)$ and its proof is like the proof of Lemma 13.

Theorem 14. *Let $n \equiv 0(mod\ 9)$. Then*
1. $(Q^*(\sqrt{\frac{n}{9}}) \setminus Q^{***}(\sqrt{\frac{n}{9}})) \cup Q^{***}(\sqrt{9n}) = Q^{'''}(\sqrt{n}) \setminus Q^*(\sqrt{n})$ and
2. $Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n}) = \{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\}$.

The following corollary is an immediate consequence of Corollary 4 and Lemma 13.

Corollary 15. *Let $n \equiv 2, 5$ or $8(mod\, 9)$. Then:*
1. $Q^{***}(\sqrt{9n}) = Q^{'''}(\sqrt{n}) \setminus Q^*(\sqrt{n})$ *and*
2. $\{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\} = Q^*(\sqrt{n})$.

Theorem 16. *Let $n \not\equiv 0(mod\, 9)$. Then*
$Q^{***}(\sqrt{9n}) \cup \{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\}$ *is an M-subset of $Q^{'''}(\sqrt{n})$.*
Proof: Let $n \equiv 1, 3, 4, 6$ or $7(mod\, 9)$. Then by Lemma 13, $Q^{***}(\sqrt{9n}) = Q^{'''}(\sqrt{n}) \setminus Q^*(\sqrt{n})$ and $\{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\} = Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$. However, if $n \equiv 2, 5$ or $8(mod\, 9)$ then,as mentioned earlier, $Q^{***}(\sqrt{n})$ is empty. Hence the above result holds for all $n \not\equiv 0(mod\, 9)$. Thus if $n \not\equiv 0(mod\, 9)$ then
$Q^{***}(\sqrt{9n}) \cup \{\frac{\alpha}{3} : \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\} = Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$.
If $n \equiv 2, 5,$ or $8(mod\, 9)$ then, by Corollary 4, $Q^{***}(\sqrt{n})$ is empty. However, if $n \equiv 1, 3, 4, 6$ or $7(mod\, 9)$ then, by Theorem 3, $Q^{***}(\sqrt{n})$ is an M-subset of $Q^{'''}(\sqrt{n})$. Also since $Q^*(\sqrt{n})$ is not M-subset so $Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ and $Q^{'''}(\sqrt{n}) \setminus Q^*(\sqrt{n})$ are not M-subsets of $Q^{'''}(\sqrt{n})$. By Theorems 2 and 3, we know that $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ is an M-subset of $Q^{'''}(\sqrt{n})$ for all $n \not\equiv 0(mod\, 9)$.
Thus $Q^{***}(\sqrt{9n}) \cup \{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\}$ is an M-subset of $Q^{'''}(\sqrt{n})$ for all $n \not\equiv 0(mod\, 9)$.
Following theorem is an extension of Theorem 16 for each non-square n and its proof follows from Theorem 14.

Theorem 17. *Let $n \equiv 0(mod\, 9)$. Then*
$Q^{***}(\sqrt{9n}) \cup \{\frac{\alpha}{3} : \alpha = \frac{3a+\sqrt{9n}}{c} \in Q^*(\sqrt{9n}) \setminus Q^{***}(\sqrt{9n})\}$ *is an M-subset of $Q^{'''}(\sqrt{n})$.*

Theorem 18. *Let $n \equiv 0(mod\, 9)$. Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ with $b = \frac{a^2-n}{c}$. Then:*
1. If $3 \nmid a$ then $\frac{\alpha}{3}$ belongs to $Q^{***}(\sqrt{9n})$.
2. If $3 \mid a$ then $\frac{\alpha}{3}$ belongs to $Q^*(\sqrt{\frac{n}{9}}) \setminus Q^{***}(\sqrt{\frac{n}{9}})$ or $Q^{***}(\sqrt{9n})$ according as $\alpha \in Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ or $Q^{***}(\sqrt{n})$.

Proof. Let $n \equiv 0(mod\, 9)$. Let $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ with $b = \frac{a^2-n}{c}$. Then:
(1) If $3 \nmid a$ then $bc = (a^2 - n) \equiv 1, 4$ or $7(mod\, 9)$ so $3 \nmid b$. Therefore, by Lemma 1(2), $\frac{\alpha}{3}$ belongs to $Q^{***}(\sqrt{9n})$.

(2) If $3 \mid a$ then $(a^2 - n) \equiv 0(mod\, 9)$. So b, c cannot be both divisible by 3, as otherwise $(a, b, c) \neq 1$. Thus exactly one of b, c is divisible by 3. Therefore, again by second part of Lemma 1, if b is not divisible by 3 then $\frac{\alpha}{3}$ belongs to $Q^{***}(\sqrt{9n})$. But if b is divisible by 3 then, from the proof of Lemma 13(2), $\frac{\alpha}{3}$ belongs to $Q^*(\sqrt{\frac{n}{9}}) \setminus Q^{***}(\sqrt{\frac{n}{9}})$. That is, $\frac{\alpha}{3}$ belongs to $Q^*(\sqrt{\frac{n}{9}}) \setminus Q^{***}(\sqrt{\frac{n}{9}})$ or $Q^{***}(\sqrt{9n})$ according as $\alpha \in Q^*(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$ or $Q^{***}(\sqrt{n})$. □

Following example illustrates the above theorem.

Example 19. *Let $n = 27$. Then $\alpha = \frac{1+\sqrt{27}}{1} \in Q^*(\sqrt{27})$ but $\frac{\alpha}{3} = \frac{1+\sqrt{27}}{3} = \frac{3+\sqrt{243}}{9} \in Q^{***}(\sqrt{243})$. Also $\beta = \frac{3+\sqrt{27}}{1} \in Q^*(\sqrt{27})$ but $\frac{\beta}{3} = \frac{1+\sqrt{3}}{1} \in Q^*(\sqrt{3}) \setminus Q^{***}(\sqrt{3})$. Similarly $\gamma = \frac{3+\sqrt{27}}{18} \in Q^{***}(\sqrt{27})$ whereas $\frac{\gamma}{3} = \frac{9+\sqrt{243}}{162} \in Q^{***}(\sqrt{243})$.*

Summarizing the above results we have the following

Theorem 20. *Let* $n \equiv 0(mod\, 9)$. *Then* $Q^{'''}(\sqrt{n}) = (Q^*(\sqrt{\frac{n}{9}}) \setminus Q^{***}(\sqrt{\frac{n}{9}})) \cup Q^*(\sqrt{n}) \cup Q^{***}(\sqrt{9n})$.

Proof. Follows from Theorems 17 and 18. □

We conclude this paper with the following observations.
If $n \equiv 2, 5$ *or* $8(mod\ 9)$, *then* $Q^{'''}(\sqrt{n})$, $Q^{'''}(\sqrt{9n}) \setminus Q^{'''}(\sqrt{n})$ *are both* M*-subsets of* $Q^{'''}(\sqrt{9n})$ *and in particular* $Q^{'''}(\sqrt{n}) \subset Q^{'''}(\sqrt{9n})$. *If* $n \equiv 1, 3, 4, 6$ *or* $7(mod\ 9)$, *then* $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$, *and* $Q^{'''}(\sqrt{9n}) \setminus Q^{'''}(\sqrt{n})$ *are all* M*-subsets of* $Q^{'''}(\sqrt{9n})$. *In particular* $Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n}) \subseteq Q^{'''}(\sqrt{9n})$. *That is* $Q^{'''}(\sqrt{9n}) \cap Q^{'''}(\sqrt{n}) = Q^{'''}(\sqrt{n}) \setminus Q^{***}(\sqrt{n})$. *For the cases* $n \not\equiv 0(mod\, 9)$. *For* $n = 2$, $9n = 18$, $Q^{***}(\sqrt{2}) = \{\}$, $Q^{'''}(\sqrt{2}) = (\sqrt{2})^M \cup (-\sqrt{2})^M$,*and* $Q^{'''}(\sqrt{18}) \setminus Q^{'''}(\sqrt{2}) = (\sqrt{18})^M \cup (-\sqrt{18})^M$. *So* $Q^{'''}(\sqrt{18})$ *has exactly 4 orbits under the action of* M. *Also if* $n = 3$, $9n = 27$, $Q^{'''}(\sqrt{3}) \setminus Q^{***}(\sqrt{3}) = (\sqrt{3})^M \cup (-\sqrt{3})^M, Q^{'''}(\sqrt{27}) \setminus Q^{'''}(\sqrt{3}) = (\sqrt{27})^M \cup (-\sqrt{27})^M$. *So* $Q^{'''}(\sqrt{27})$ *has exactly 4 orbits under the action of* M. *Similarly if* $n = 5$, $9n = 45$, $Q^{'''}(\sqrt{5}) = (\sqrt{5})^M \cup (-\sqrt{5})^M \cup (\frac{1+\sqrt{5}}{2})^M \cup (\frac{1-\sqrt{5}}{2})^M$, $(\frac{1+\sqrt{45}}{2})^M \cup (\frac{1-\sqrt{45}}{2})^M \cup (\sqrt{45})^M \cup (-\sqrt{45})^M$. *So* $Q^{'''}(\sqrt{45})$ *splits into exactly 8 orbits under the action of* M.

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