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Local convergence for a family of third order methods in Banach spaces

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Abstract. We present a local convergence analysis of a family of third order methods for approximating a locally unique solution of nonlinear equations in a Banach space setting. Recently, the semilocal convergence analysis of this method was studied by Chun, Stănică and Neta in [10]. These authors extended earlier results by Kou, Li [17] and others [8, **?**, 11, 13, 14]. The convergence analysis is based on hypotheses up to the second Fréchet derivative of the operator involved. This work further extends the results of [10] and provides computable convergence ball and computable error bounds under hypotheses only up to the first Fréchet derivative.

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1. Introduction

In this study, we are concerned with the problems of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$\mathcal{F}(x) = 0 \tag{1.1}$$

where $\mathcal{F}$ is a Fréchet-differentiable operator defined on a convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Using mathematical modelling, many problems in computational sciences and other disciplines can be brought in a form like (1.1) [2, 3, 6, 9, 18, 19, 20, 23]. The solutions of these equations (1.1) can rarely be found in closed form. Therefore solutions of these equations (1.1) are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for finding such solutions is essentially connected to Newton-like methods [1–22]. The study about convergence of iterative procedures is normally centered on two types: semilocal and local convergence analysis. The semilocal convergence analysis is based on the information around an initial point to give criteria ensuring the convergence of iterative procedures. While the local analysis is based on the information around a solution to find estimates of the radii of convergence balls. There exist many studies which deal with the local and the semilocal convergence analysis of Newton-like methods such as [1–22].

Majorizing sequences have been used extensively in connection to the Kantorovich theorem when studying the convergence of these methods [2, 3, 4, 5, 6, 18, 19, 20]. Candela and Marquina [8, **?**], Parida and Gupta [19], Ezquerro and Hernández [11], Gutiérrez and Hernández [13, 14], Argyros [2, 3, 4, 5, 6] used this idea for several high-order methods. In particular, Kou and Li [17] introduced a third order family of methods for solving equation (1.1), when $\mathbf{X} = \mathbf{Y} = \mathbb{R}$ defined by

$$\left.\begin{aligned} y_n &= x_n - \theta \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= x_n - \frac{\theta^2+\theta-1}{\theta^2}\mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) - \frac{1}{\theta^2}\mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n) \\ &\text{for each} \quad n = 0, 1, 2, \dots \end{aligned}\right\} \tag{1.2}$$

where x_0 is an initial point and $\theta \in \mathbb{R}\backslash\{0\}$. This family uses two evaluations of $\mathcal{F}$ and one evaluation of $\mathcal{F}'$. Third order methods requiring one evaluation of $\mathcal{F}$ and two evaluations of $\mathcal{F}'$ can be found in [2, 6, 10, 17]. It is well known that the convergence domain of high order methods is small [1, 2, 6, 13, 16, 20]. This fact limits the applicability of these methods. In the present study, we are motivated by this fact and the recent work by Chun, Stănică and Neta [10] where a semilocal convergence analysis of the third order method (1.2) in a Banach space setting is presented. Their convergence conditions require hypotheses up to the second Fréchet derivative. Hence, their results cannot apply when operator $\mathcal{F}$ is not twice Fréchet-differentiable on $\mathbf{D}$. In the present study, we require hypotheses up to the first Fréchet derivative of operator $\mathcal{F}$. Hence, the applicability of method (1.2) is extended under our approach. Moreover, we provide a local convergence analysis that includes a computable convergence ball and error bounds which are not given in the earlier studies [8, **?**, 10, 11, 13, 14, 15, 17, 23].

The rest of the paper is organized as follows. In Section 2, we present the local convergence analysis for the third order method (1.2). The numerical examples are given in the concluding Section 3.

2. Local Convergence

Let $\mathbf{U}(w,\rho)$ and $\overline{\mathbf{U}}(w,\rho)$ stand, respectively, for the open and closed balls in $\mathbf{X}$ centered at $w \in \mathbf{X}$ and radius $\rho > 0$. Let also $\mathcal{L}(\mathbf{X},\mathbf{Y})$ stand for the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$. We present the local convergence analysis of the third order method (1.2) under the conditions (**C**):

C_1: $\mathcal{F} : \mathbf{D} \subset \mathbf{X} \longrightarrow \mathbf{Y}$ is Fréchet differentiable and there exists $x^\star \in \mathbf{D}$ such that $\mathcal{F}(x^\star) = 0$ and $\mathcal{F}'(x^\star)^{-1} \in \mathcal{L}(\mathbf{Y},\mathbf{X})$;

C_2: operator $\mathcal{F}'$ satisfies the Lipschitz condition

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(y))\right\| \le \mathcal{K}\,\|x-y\| \quad \text{for each} \quad x, y \in \mathbf{D};$$

C_3: operator $\mathcal{F}'$ satisfies the center-Lipschitz condition

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\star))\right\| \le \mathcal{K}_0\,\|x-x^\star\| \quad \text{for each} \quad x \in \mathbf{D};$$

C_4: $\left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x)\right\| \le \mathsf{L}$ for each $x \in \mathbf{D}$;

C_5: there exists $\theta \in \mathbb{R} - \{0\}$ such that

$$\mathsf{L}|1-\theta| < 1;$$

C_6: there exists $r > 0$ and $\theta \in \mathbb{R} - \{0\}$ such that

$$r \leq r_0 := \frac{1 - \mathsf{L}|1-\theta|}{\mathcal{K}_0 + \dfrac{\mathcal{K}}{2}}$$

and

$$f_\theta(r) > 0$$

where

$$f_\theta(t) = \mathcal{K}_0\Big(\mathcal{K}_0 + \frac{\mathcal{K}}{2}\Big)t^2 - \Big(2\mathcal{K}_0 - \mathcal{K}_0\mathsf{L}\frac{|\theta-1|}{\theta^2} + \frac{\mathcal{K}}{2} + \frac{\mathcal{K}\mathsf{L}}{2\theta^2}\Big)t + 1 - \mathsf{L}\frac{|\theta-1|}{\theta^2} - \frac{\mathsf{L}^2}{\theta^2}|1-\theta|;$$

C_7: $\mathbf{U}(x^\star, r) \subset \mathbf{D}$.

It is worth noticing that condition C_2 always implies C_3 but not necessarily vice versa. We also have that

$$\mathcal{K}_0 \leq \mathcal{K}$$

holds in general and K/K_0 can be arbitrarily large [2,6]. In practice the computation of constant $\mathcal{K}$ requires the computation of $\mathcal{K}_0$ as a special case. Condition C_3 is used to find tighter upper bounds on the norms $\left\|\mathcal{F}'(x)^{-1}\mathcal{F}'(x^\star)\right\|$ than if only condition C_2 is used (provided that $\mathcal{K}_0 < \mathcal{K}$ see (2.5) and (2.6)).

Let $x_0 \in \mathbf{D}$ be fixed. Then it follows from condition C_2 that

C_2' : Operator $\mathcal{F}'$ satisfies the center-Lipschitz condition

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x_0))\right\| \leq \overline{\mathcal{K}_0}\,\|x - x_0\| \quad \text{for each} \quad x \in \mathbf{D}.$$

Notice again that

$$\overline{\mathcal{K}_0} \leq \mathcal{K}$$

and $K/\overline{\mathcal{K}_0}$ can be arbitrarily large [2,6].

Later in the proof of Theorem 1 using condition C_2' instead of condition C_2 leads to a tighter error estimate for the upper bounds on $\|y_0 - x^\star\|$ and $\|x_1 - x^\star\|$ than if only condition C_2 is used (see (2.3) and (2.6) for $n = 0$).

Next we show the main local convergence result for the third order method (1.2) under the (**C**) conditions.

Theorem 1. *Suppose that the (**C**) conditions hold. Then, sequence $\{x_n\}$ generated by the third order method (1.2) for $x_0 \in \mathbf{U}(x^\star, r) - \{x^\star\}$ is well defined, remains in $\mathbf{U}(x^\star, r)$ for each $n = 0, 1, 2, 3, \ldots$ and converges to $x^\star$. Moreover the following estimates hold for each $n = 0, 1, 2, \ldots$*

$$\begin{aligned}\|y_n - x^\star\| &\leq \frac{1}{1 - \mathcal{K}_0\,\|x_n - x^\star\|}\Big[\frac{\overline{\mathcal{K}}}{2}\,\|x_n - x^\star\| + \mathsf{L}|1-\theta|\Big]\,\|x_n - x^\star\| \\ &\leq \|x_n - x^\star\| < r \end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\|x_{n+1} - x^\star\| &\leq \frac{1}{1 - \mathcal{K}_0\,\|x_n - x^\star\|}\Big[\frac{\overline{\mathcal{K}}}{2}\,\|x_n - x^\star\|^2 + \frac{|\theta-1|}{\theta^2}\mathsf{L}\,\|x_n - x^\star\| \\ &\quad + \frac{1}{\theta^2}\mathsf{L}\,\|y_n - x^\star\|\Big] \\ &< \|x_n - x^\star\| < r \end{aligned} \tag{2.2}$$

where

$$\overline{\mathcal{K}} = \begin{cases} \overline{\mathcal{K}_0}, & \textit{if} \quad n = 0 \\ \mathcal{K}, & \textit{if} \quad n > 0 \end{cases}$$

Proof. By the hypothesis we have that $x_0 \in \mathbf{U}(x^\star, r) - \{x^\star\}$. Then, using (**C**$_3$) and (**C**$_5$) we get that

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(x^\star))\right\| \leq \mathcal{K}_0 \left\|x_0 - x^\star\right\| < \mathcal{K}_0 r < 1. \tag{2.3}$$

It follows from (2.3) and the Banach lemma on invertible operators [2, 6, 16, 18, 20] that $\mathcal{F}'(x_0)^{-1} \in \mathcal{L}(\mathbf{Y}, \mathbf{X})$ and

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \leq \frac{1}{1 - \mathcal{K}_0 \left\|x_0 - x^\star\right\|}. \tag{2.4}$$

Then, y_0 and y_1 are well defined. Using the first substep in (1.2) for $n = 0$ and (2.4) we obtain in turn that

$$\begin{aligned} y_0 - x^\star &= x_0 - x^\star - \theta\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0) \\ &= \mathcal{F}'(x_0)^{-1}\Big[\int_0^1 \Big(\mathcal{F}'(x_0 + \tau(x^\star - x_0)) - \mathcal{F}'(x_0)\Big)(x^\star - x_0)\mathrm{d}\tau \\ &\qquad + (\theta - 1)\int_0^1 \mathcal{F}'(x_0 + \tau(x^\star - x_0))(x^\star - x_0)\mathrm{d}\tau\Big]. \end{aligned} \tag{2.5}$$

Hence, using (2.5), (**C**$_3$), (**C**$_4$), (2.4) and (**C**$_5$) we obtain that

$$\begin{aligned} \|y_0 - x^\star\| &\leq \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \Big[\Big\|\mathcal{F}'(x^\star)^{-1}\int_0^1 \Big(\mathcal{F}'(x_0 + \tau(x^\star - x_0)) - \mathcal{F}'(x_0)\Big) \\ &\qquad \times (x^\star - x_0)\mathrm{d}\tau + |1 - \theta| \Big\|\mathcal{F}'(x^\star)^{-1}\int_0^1 \mathcal{F}'(x_0 + \tau(x^\star - x^\star))(x^\star - x_0)\mathrm{d}\tau\Big\|\Big] \\ &\leq \frac{1}{1 - \mathcal{K}_0 \|x_0 - x^\star\|}\Big[\frac{\overline{\mathcal{K}_0}}{2}\|x_0 - x^\star\| + \mathsf{L}\,|1 - \theta|\Big]\|x_0 - x^\star\| \\ &\leq \frac{1}{1 - \mathcal{K}_0 r}\Big[\frac{\overline{\mathcal{K}_0}}{2}r + \mathsf{L}\,|1 - \theta|\Big]\|x_0 - x^\star\| \leq \|x_0 - x^\star\| < r, \end{aligned}$$

which shows $y_0 \in \mathbf{U}(x^\star, r)$ and (2.1) holds for $n = 0$. Then, using the second substep in (1.2) for $n = 0$ we have that

$$\begin{aligned} x_1 - x^\star &= x_0 - x^\star - \frac{\theta^2 + \theta - 1}{\theta^2}\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0) - \frac{1}{\theta^2}\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_0) \\ &= \mathcal{F}'(x_0)^{-1}\Big[\int_0^1 \Big(\mathcal{F}'(x_0 + \tau(x^\star - x_0)) - \mathcal{F}'(x_0)\Big)(x^\star - x_0)\mathrm{d}\tau \\ &\qquad + \frac{\theta - 1}{\theta^2}\int_0^1 \mathcal{F}'(x^\star + \tau(x_0 - x^\star))(x_0 - x^\star)\mathrm{d}\tau \\ &\qquad + \frac{1}{\theta^2}\int_0^1 \mathcal{F}'(x^\star + \tau(y_0 - x^\star))(y_0 - x^\star)\mathrm{d}\tau\Big]. \end{aligned} \tag{2.6}$$

Using (2.6), ($\mathbf{C}_3$), ($\mathbf{C}_4$) and the second condition in ($\mathbf{C}_6$) we get that

$$\begin{aligned}\|x_1 - x^\star\| &\leq \frac{1}{1-\mathcal{K}_0\|x_0-x^\star\|}\Big[\frac{\overline{\mathcal{K}_0}}{2}\|x_0-x^\star\|^2 + \frac{|\theta-1|}{\theta^2}\mathsf{L}\,\|x_0-x^\star\| + \frac{\mathsf{L}}{\theta^2}\|y_0-x^\star\|\Big]\\ &\leq \frac{1}{1-\mathcal{K}_0\|x_0-x^\star\|}\Big[\frac{\overline{\mathcal{K}_0}}{2}\|x_0-x^\star\| + \mathsf{L}\frac{|\theta-1|}{\theta^2} + \frac{\mathsf{L}}{\theta^2}\Big(\frac{1}{1-\mathcal{K}_0\|x_0-x^\star\|}\\ &\quad\Big(\frac{\mathcal{K}_0}{2}\|x_0-x^\star\| + \mathsf{L}\,|1-\theta|\Big)\Big)\Big]\|x_0-x^\star\|\\ &< \|x_0-x^\star\| < r,\end{aligned}$$

which shows that $y_1 \in \mathbf{U}(x^\star, r)$ and that (2.2) holds for $n = 0$. Suppose that (2.1), (2.2), $y_k, s_{k+1} \in \mathbf{U}(x^\star, r)$ hold for all $k \leq n-1$. Then, y_n and x_{n+1} are well defined. As in (2.3), we also have that

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x_n) - \mathcal{F}(x^\star))\right\| \leq \mathcal{K}_0\|x_n - x^\star\| < \mathcal{K}_0 r < 1.$$

Hence, $\mathcal{F}'(x_n)^{-1} \in \mathcal{L}(\mathbf{Y}, \mathbf{X})$ and

$$\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x^\star)\right\| \leq \frac{1}{1-\mathcal{K}_0\|x_n-x^\star\|}. \tag{2.7}$$

Using the fist step in (1.2) we get as in (2.5) that

$$\begin{aligned}y_n - x^\star = \mathcal{F}'(x_n)^{-1}\Big[\int_0^1 (\mathcal{F}'(x_n+\tau(x^\star-x_n)) - \mathcal{F}'(x_n))(x^\star-x_n)\mathrm{d}\tau\\ + (\theta-1)\int_0^1 \mathcal{F}'(x_n+\tau(x^\star-x_n))(x^\star-x_n)\mathrm{d}\tau\Big]\end{aligned}$$

and

$$\begin{aligned}\|y_n - x^\star\| &\leq \frac{1}{1-\mathcal{K}_0\|x_n-x^\star\|}\Big[\frac{\overline{\mathcal{K}}}{2}\|x_n-x^\star\| + \mathsf{L}\,|1-\theta|\Big]\|x_n-x^\star\|\\ &\leq \frac{1}{1-\mathcal{K}_0 r}\Big[\frac{\overline{\mathcal{K}}}{2}r + \mathsf{L}\,|1-\theta|\Big]\|x_n-x^\star\| \leq \|x_n-x^\star\| < r,\end{aligned}$$

which shows that $y_n \in \mathbf{U}(x^\star, r)$ and that (2.1) holds. Moreover, from the second step in (1.2) as in (2.6) we get that

$$\begin{aligned}x_{n+1} - x^\star = \mathcal{F}'(x_n)^{-1}\Big[\int_0^1 (\mathcal{F}'(x_n+\tau(x^\star-x_n)) - \mathcal{F}'(x_n))(x^\star-x_n)\mathrm{d}\tau\\ \frac{\theta-1}{\theta^2}\int_0^1 \mathcal{F}'(x^\star+\tau(x_n-x^\star))(x_n-x^\star)\mathrm{d}\tau + \frac{1}{\theta^2}\int_0^1 \mathcal{F}'(x^\star+\tau(y_n-x^\star))(y_n-x^\star)\mathrm{d}\tau\Big]\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-x^\star\| &\le \frac{1}{1-\mathcal{K}_0\|x_n-x^\star\|}\Big[\frac{\overline{\mathcal{K}}}{2}\|x_n-x^\star\|+\frac{|\theta-1|}{\theta^2}\mathsf{L}\|x_n-x^\star\| \\ &\quad+\frac{1}{\theta^2}\mathsf{L}\|y_n-x^\star\|\Big] \\ &\le \frac{1}{1-\mathcal{K}_0\|x_n-x^\star\|}\Big[\frac{\overline{\mathcal{K}}}{2}\|x_n-x^\star\|+\frac{|\theta-1|}{\theta^2}\mathsf{L}+\frac{\mathsf{L}}{\theta^2}\Big(\frac{1}{1-\mathcal{K}_0\|x_n-x^\star\|} \\ &\quad\Big(\frac{\overline{\mathcal{K}}}{2}\|x_n-x^\star\|+\mathsf{L}|1-\theta|\Big)\Big)\Big]\|x_n-x^\star\| \\ &< \|x_n-x^\star\|<r,\end{aligned}$$

which shows that $x_{n+1}\in\mathbf{U}(x^\star,r)$ and that (2.2) holds. The induction is completed. Then, it follows from $\|x_{n+1}-x^\star\|<\|x_n-x^\star\|$ that $\lim_{n\to\infty}x_n=x^\star$. □

Remark 2.

(1) In view of $(\mathbf{C}_3)$ and the estimate

$$\begin{aligned}\big\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x)\big\| &= \big\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x)-\mathcal{F}'(x^\star))+\mathcal{I}\big\| \\ &\le 1+\big\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x)-\mathcal{F}'(x^\star))\big\|\le 1+\mathcal{K}_0\|x-x^\star\| \end{aligned}\tag{2.8}$$

condition $(\mathbf{C}_4)$ can be dropped and L – in $(\mathbf{C}_5)$ and in $(\mathbf{C}_6)$ – can be replaced by

$$\mathsf{L}(r)=1+\mathcal{K}_0 r. \tag{2.9}$$

(2) In practice we shall choose θ (see numerical examples) so that the two conditions in $(\mathbf{C}_6)$ hold that is the radius r exists. There exist such cases. Let us list one:

Suppose that

$$\frac{\mathsf{L}^2}{\theta^2}|1-\theta|+\mathsf{L}\frac{|\theta-1|}{\theta^2}-1>0.$$

Then, polynomial f_θ has a unique positive root r_θ. If $(\mathbf{C}_5)$ holds and $r_\theta<r_0$ (or $f_\theta(r_0)>0$) then, we can choose $r=r_0$. In view of $(\mathbf{C}_4)$, we have that $\mathsf{L}\ge 1$.

It follows from $(\mathbf{C}_5)$ and $\mathsf{L}\ge 1$ (see also the numerical examples) that we should only choose $\theta\in(0,2)$ for our conditions to work although the convergence of the third order method (1.2) may be possible for $\theta\in\mathbb{R}\backslash[0,2]$

(3) It is worth noticing that the earlier results [8, **?**, 10, 11, 13, 14, 17, 23] use hypotheses on the second Fréhet derivative (or higher) for the semilocal convergence of the third order method (1.2). In this study we use only hypotheses on the first Fréchet derivative. In the local case the earlier works do not provide a computable convergence ball or computable error bounds based on Lipschitz or other constants.

(4) The results obtained here can be used for operators $\mathcal{F}$ satisfying autonomous differential equations [2, 6, 16, 18] of the form

$$\mathcal{F}'(x)=\mathcal{P}(\mathcal{F}(x))$$

where $\mathcal{P}$ is a continuous operator. Then, since $\mathcal{F}'(x^\star)=\mathcal{P}(\mathcal{F}(x^\star))=\mathcal{P}(0)$, we can apply the results without actually knowing $x^\star$. For example, let $\mathcal{F}(x)=e^x-1$. Then, we can choose: $\mathcal{P}(x)=x+1$.

(5) The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods

and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refineent strategies [2, 6, 16, 18].

(6) In view of (**C**$_5$) and the first inequality in (**C**$_6$) the radius r is such that

$$r \leq r_{\mathcal{A}} = \frac{1}{\mathcal{K}_0 + \dfrac{\mathcal{K}}{2}}. \tag{2.10}$$

The parameter $r_{\mathcal{A}}$ was shown by us to be the convergence radius of Newton's method [2, 6]

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad \text{for each} \quad n = 0, 1, 2, \ldots \tag{2.11}$$

under the conditions (**C**$_1$)–(**C**$_3$). It follows from (2.10) that the convergence radius r of the third order method (1.2) cannot be larger than the convergence radius $r_{\mathcal{A}}$ of the second order Newton's method (2.11). As already noted in [2, 4, 6] $r_{\mathcal{A}}$ is at least as large as the convergence ball given by Rheinboldt [22]

$$r_{\mathcal{R}} = \frac{2}{3\,\mathcal{K}}.$$

In particular, for $\mathcal{K}_0 < \mathcal{K}$ we have that

$$r_{\mathcal{R}} < r_{\mathcal{A}}$$

and

$$\frac{r_{\mathcal{R}}}{r_{\mathcal{A}}} = \frac{2\mathcal{K}_0/\mathcal{K} + 1}{3} \longrightarrow \frac{1}{3} \quad \text{as} \quad \frac{\mathcal{K}_0}{\mathcal{K}} \longrightarrow 0.$$

That is our convergence ball $r_{\mathcal{A}}$ is at most three times larger than Rheinboldt's. The same value for $r_{\mathcal{R}}$ was also given by Traub [23].

3. Numerical Examples

For $\mathbf{X} = \mathbf{Y} = \mathbb{R}^m$, the method (1.2) yields

$$\left.\begin{aligned} \mathcal{F}'(\mathbf{x}_n)\mathbf{p}_n &= \theta\mathcal{F}(\mathbf{x}_n), \quad \mathbf{y}_n = \mathbf{x}_n - \mathbf{p}_n \\ \mathcal{F}'(\mathbf{x}_n)\mathbf{q}_n &= \frac{(\theta^2+\theta-1)\mathcal{F}(\mathbf{x}_n) + \mathcal{F}(\mathbf{y}_n)}{\theta^2}, \quad \mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{q}_n. \end{aligned}\right\}$$

We present three numerical examples in this section.

Example 3. *Let* $\mathbf{X} = \mathbf{Y} = \mathbb{R}^3$, $\mathbf{D} = \overline{\mathbf{U}}(0,1)$ *and* $x^\star = (0,0,0)$. *We define function* $\mathcal{F}$ *on* $\mathbf{D}$ *as*

$$\mathcal{F}(x,y,z) = \Big(e^x - 1, \frac{e-1}{2}\,y^2 + y, z\Big). \tag{3.1}$$

Then, the Fréchet derivative of $\mathcal{F}$ *is given by*

$$\mathcal{F}'(x,y,z) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & (e-1)\,y+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that we have:

$$\left.\begin{aligned} \mathcal{F}(x^\star) = 0, \quad \mathcal{F}'(x^\star) = \mathcal{F}'(x^\star)^{-1} = \mathrm{diag}\,\{1,1,1\} \\ \mathsf{L} = e, \quad \mathcal{K}_0 = e-1, \quad \overline{\mathcal{K}_0} = \mathcal{K} = e. \end{aligned}\right\}$$

To ascertain the convergence-order of the method (1.2)*, we use the concept of computational order of convergence (**COC**)* [6, 11]

$$\rho = \sup \frac{\ln\left(\dfrac{\|\mathbf{x}_{n+2}-\mathbf{x}_{n+1}\|}{\|\mathbf{x}_{n+1}-\mathbf{x}_{n}\|}\right)}{\ln\left(\dfrac{\|\mathbf{x}_{n+1}-\mathbf{x}_{n}\|}{\|\mathbf{x}_{n}-\mathbf{x}_{n-1}\|}\right)} \quad \textit{for} \quad n \in \mathbb{N}_{>0}.$$

We solve the nonlinear system (3.1) *by the third order method* (1.2) *for* $\mathbf{x_0} = (0.1, 0.1, 0.1)^T$*. Notive that* $x_0 \in \mathbf{U}(x^\star, r)$*. Results of our computation are reported in the Table 1.*

n	$\|\mathbf{x}_n - \mathbf{x}_{n-1}\|_2$	$\|\mathcal{F}(\mathbf{x})\|_2$
0	$---$	0.181254010020148
1	0.172349059098655	0.001036567529705
2	0.001129080546855	0.000000001633261
3	0.000000001633894	0.000000000000000

TABLE 1. Solving (3.1) by the third order method (1.2) for $\mathbf{x_0} = (0.1, 0.1, 0.1)^{\mathrm{T}}$.

In the Table 1, we notice that $\rho \approx 2.87415$*. For* $\theta = 1.2$ *the condition (*C_5*) yields*

$$\mathsf{L}\,|1-\theta| = 0.3329311881 < 1$$

and the condition (C_6*) yields*

$$r_0 \approx 0.1482876006 \quad \textit{and} \quad f_1(t) > 0 \quad \forall t \in (0, 0.3329311881).$$

Thus our conditions (C_5*) and (*C_6*) hold. Thus our results are applicable for analysing convergence of the method* (1.2)*.*

Example 4. *Let* $\mathbf{X} = \mathbf{Y} = \mathbb{C}[0,1]$*, the space of continuous functions defined on* $[0,1]$ *be equipped with the max norm and* $\mathbf{D} = \overline{\mathbf{U}}(0,1)$*. Define function* $\mathcal{F}$ *on* $\mathbf{D}$ *by*

$$\mathcal{F}(h)(x) = h(x) - 5\int_0^1 x\,\theta\,h(\theta)^3\,d\theta. \tag{3.2}$$

Then, the Fréchet derivative of $\mathcal{F}$ *is given by*

$$\mathcal{F}'(h[u])(x) = u(x) - 15\int_0^1 x\,\theta\,h(\theta)^2\,u(\theta)\,d\theta \quad \text{for all } u \in \mathbf{D}.$$

Some algebraic manipulations yield

$$\mathsf{L} = \mathsf{L}(r) = 1 + 7.5\,r, \quad \mathcal{K}_0 = 7.5 \quad \textit{and} \quad \overline{\mathcal{K}_0} = \mathcal{K} = 15.$$

For $\theta = 1$*, we obtain* $r_0 = 0.06666\ldots$ *and* $r_1 = 0.035726559$*. Thus we must choose* $r \in (0, r_1)$*.*

Example 5. *Let* $\mathbf{X} = \mathbf{Y} = \mathbb{R}^{m-1}$ *for natural integer* $n \geq 2$*.* $\mathbf{X}$ *and* $\mathbf{Y}$ *are equipped with the max-norm* $\|\mathbf{x}\| = \max_{1\leq i\leq n-1} \|x_i\|$*. The corresponding matrix norm is*

$$\|A\| = \max_{1\leq i\leq m-1} \sum_{j=1}^{j=m-1} |a_{ij}|$$

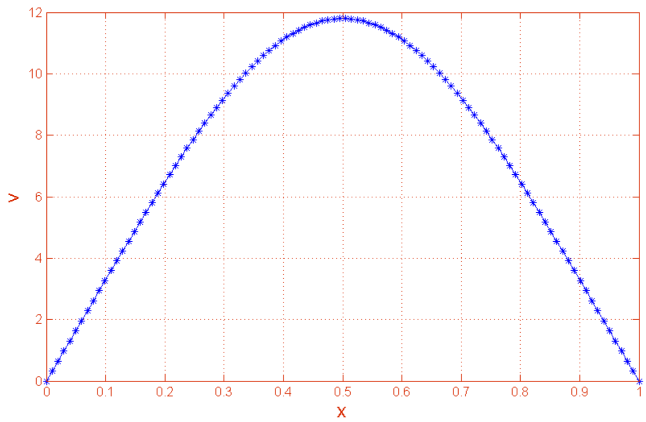

FIGURE 1. Solution of the boundary value problem (3.3).

for $A = (a_{ij})_{1\le i,j\le m-1}$. *On the interval* $[0,1]$, *we consider the following two point boundary value problem*

$$\begin{cases} v'' + v^2 = 0 \\ v(0) = v(1) = 0. \end{cases} \tag{3.3}$$

[**?**, see]*12,6. To discretize the above equation, we divide the interval* $[0,1]$ *into* m *equal parts with length of each part:* $h = 1/m$ *and coordinate of each point:* $x_i = i\,h$ *with* $i = 0,1,2,\ldots,m$. *A second-order finite difference discretization of equation* (3.3) *results in the following set of nonlinear equations*

$$\mathcal{F}(\mathbf{v}) := \begin{cases} v_{i-1} + h^2\, v_i^2 - 2v_i + v_{i+1} = 0 \\ \text{for} \quad i = 1,2,\ldots,(m-1) \quad \text{and from (3.3)} \quad v_0 = v_m = 0 \end{cases} \tag{3.4}$$

where $\mathbf{v} = [v_1, v_2, \ldots, v_{(m-1)}]^T$ *For the above system-of-nonlinear-equations, we provide the Fréchet derivative*

$$\mathcal{F}'(\mathbf{v}) = \begin{bmatrix} \frac{2v_1}{m^2} - 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{2v_2}{m^2} - 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \frac{2v_3}{m^2} - 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{2v_{(m-1)}}{m^2} - 2 \end{bmatrix}.$$

Let $m = 101$, $x_0 = [5,5,\ldots,5]^T$ *and we choose* $\theta = 1$. *To solve the linear systems (step 1 and step 2 in the method* (1.2)*), we employ MatLab routine "linsolve" which uses LU factorization with partial pivoting. Figure 1 plots our numerical solution.*

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