Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 53(2)(2021) pp. 19-33

A Study of Completely Inverse Paramedial AG-Groupoids

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Received: 25 Feburary, 2020 / Accepted: 20 January, 2021 / Published online: 22 Feburary, 2021

Abstract.: A magma S that meets the identity, $xy\cdot z = zy\cdot x$, $\forall x, y, z \in S$ is called an AG-groupoid. An AG-groupoid S gratifying the paramedial law: $uv \cdot wx = xv \cdot wu$, $\forall\, u, v, w, x \in S$ is called a paramedial AG-groupoid. Every AG-grouoid with a left identity is paramedial. We extend the concept of inverse AG-groupoid [4, 7] to paramedial AG-groupoid and investigate various of its properties. We prove that inverses of elements in an inverse paramedial AG-groupoid are unique. Further, we initiate and investigate the notions of congruences, partial order and compatible partial orders for inverse paramedial AG-groupoid and strengthen this idea further to a completely inverse paramedial AG-groupoid. Furthermore, we introduce and characterize some congruences on completely inverse paramedial AG-groupoids and introduce and characterize the concept of separative and completely separative ordered, normal sub-groupoid, pseudo normal congruence pair, and normal congruence pair for the class of completely inverse paramedial AG-groupoids. We also provide a variety of examples and counterexamples for justification of the produced results.

AMS (MOS) Subject Classification Codes: 20N02; 08A30
Key Words: Completely inverse AG-groupoids, paramedial AG-groupoids, congruences, natural partial order.

1. Introduction

The theory of AG-groupoid is introduced in 1972 by Kazim and Naseer [6]. AG-groupoids generalize the class of commutative semigroups and satisfies the medial law, $ab \cdot cd = ac \cdot bd$. Throughout this article, S will represent an AG-groupoid otherwise stated else. This structure is closely related to a commutative semigroup because a commutative AG-groupoid is always associative [7]. An AG-groupoid may or may not contains a left identity element, and if an AG-groupoid contains a left identity, then this left identity is unique. It is important to mention here that if an AG-groupoid contains identity or even a right identity element, then it becomes a commutative monoid. Further, the left identity of an AG-groupoid permits inverses of elements in the structure. An AG-groupoid with the left identity is called AG-monoid, and satisfies the paramedial property, $ab \cdot cd = db \cdot ca$. Every paramedial AG-groupoid also satisfies the bi-commutative property,

$$ab \cdot cd = dc \cdot ba \,\forall, a, b, c, d.$$

AG-groupoid S with the property $ab \cdot c = b \cdot ac$ is called AG* and is called AG** if it satisfies the identity $a \cdot bc = b \cdot ac$. We shall use the juxtaposition to avoid excessive parenthesization and dots i.e. uv will mean $u \cdot v$, $uv \cdot wt$ for $(u \cdot v)(w \cdot t)$, and $(uv \cdot w)t$ for $((u \cdot v)w)t$. AG-groupoid is a non-associative structure in general that possess a variety of applications in the field of flock theory, geometry and finite mathematics [10, 11, 12, 13]. Fuzzification of the field has made it more interesting and applicable [1, 5, 14, 15].

Various other aspects of the said structure are also investigated by different researchers in a variety of papers [16, 17, 18, 19, 20] and the references therein. Inverse and completely inverse AG-groupoids are defined by Mushtaq and Iqbal [7], Peter V. Protic [3] and Wieslaw A. Dudek and Roman S. Gigon [4]. Some congruences on an inverse and completely inverse AG**-groupoids are defined [2, 3, 4, 8, 21]. In this section we define some congruences on completely inverse paramedial AG-groupoid. To proceed further, we start with the following definition.

Definition 1.1. [4, 7] *An AG-groupoid S is called inverse AG-groupoid, if for every $u \in S$, there exists $u' \in S$ such that $u = uu' \cdot u$ and $u'u \cdot u' = u'$. By u' we mean the inverse of u. An AG-groupoid S is called completely inverse AG-groupoid if it satisfies the identity $uu' = u'u$ for all $u \in S$.*
It is proved by Q. Mushtaq and M. Iqbal [7] *that if u' is an inverse of u and $v'$is an inverse of v in an AG-groupoid, then*

$$(uv)' = u'v'. \tag{1. 1}$$

Example 1.2. *Consider AG-groupoid $S = \{1, 2, 3, 4\}$ defined in Table 1. Then, the relation $\leq$ define as $a \leq b \iff a = aa^{-1} \cdot b$ is compatible on AG-groupoid S.*

·	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	4	3	2	1
4	3	4	1	2

Table 1

Definition 1.3. *An AG-groupoid $(S, \cdot)$ is called an ordered AG-groupoid, if S posses an order. In this case, we can write $(S, \cdot, \leq)$.*

Definition 1.4. *An ordered AG-groupoid $(S, \cdot, \leq)$ is called separative if*

(1) $\forall u, v \in S, u^2 \leq uv,\ vu \leq v^2 \Rightarrow u \leq v.$
(2) $\forall u, v \in S, u^2 \leq vu,\ uv \leq v^2 \Rightarrow u \leq v.$

In Example (1.2) the relation, $\leq$ is separative.

Definition 1.5. *A separative ordered AG-groupoid S is called completely separative if*

$$u, v, x, y \in S, x \leq y,\ (xy)u \leq (xy)v \Rightarrow x^2u \leq x^2v,\ y^2u \leq y^2v.$$

2. CONGRUENCES

In this section we define some relations on paramedial and inverse paramedial AG-groupoid S. We prove that the following relations are congruences on paramedial and inverse paramedial AG-groupoid S.

(1) $\eta = \{(u, v) \in S \times S : (\exists l \in E(S)),\ lu = lv\}$;
(2) $\mu = \{(u, v) \in S \times S : xu = xv, \forall x \in S\}$;
(3) $\rho = \{(u, v) \in S \times S : u^{-1}u = v^{-1}v\}$.

Here $E(S)$ denotes the set of idempotent elements of S.

Remark 2.1. *Let S be a paramedial AG-groupoid, and $g_1, g_2 \in E(S)$. Then by paramedial and medial law,*

$$g_1g_2 = g_1g_1 \cdot g_2g_2 = g_2g_1 \cdot g_2g_1 = g_2g_2 \cdot g_1g_1 = g_2g_1.$$

It follows that $E(S)$ is a semilattice.

The inverses in an inverse paramedial AG-grupoid are unique as proved in the following.

Remark 2.2. *Let S be an inverse paramedial AG-groupoid, and $a, b \in V(u)$. Then*

$$\begin{aligned} ua &= (ub \cdot u)a = au \cdot ub \quad \text{(by the left invertive law)} \\ &= bu \cdot ua \quad \text{(by the paramedial property)} \\ &= (ua \cdot u)b = ub \quad \text{(by the left invertive law)} \\ \Rightarrow ua &= ub. \end{aligned} \tag{2. 2}$$

Thus

$$\begin{aligned} a &= au \cdot a = (au(au \cdot a)) \quad \text{(by medial law)} \\ &= (a \cdot au)(ua) \quad \text{(by medial law)} \\ &= (a \cdot au)(ub) \quad \text{(by 2.2)} \\ &= (b \cdot au)(ua) \quad \text{(by paramedial property)} \\ &= (b \cdot au)(ub) \quad \text{(by 2.2)} \\ &= (bu)(au \cdot b) \quad \text{(by medial law)} \\ &= (bu)(bu \cdot a) \quad \text{(by left invertive law)} \\ &= (b \cdot bu)(ua) \quad \text{(by medial law)} \\ &= (b \cdot bu)(ub) \quad \text{(by 2.2)} \\ &= (bu)(bu \cdot b) \quad \text{(by medial law)} \\ &= bu \cdot b = b. \\ \Rightarrow a &= b. \end{aligned}$$

It follows that $\mid V(u) \mid= 1$, and the inverse of $u \in S$ is unique. We shall denote it by u^{-1}.

Theorem 2.3. *Let S be a paramedial AG-groupoid and $E(S) \neq \emptyset$. Then the relation η defined on S in Section 2 Part (1) is a congruence relation on S.*

Proof. Clearly, the relation η is a reflexive and symmetric on $E(S) \neq \emptyset$. In order to prove transitivity of η let $u\eta v, v\eta w$. Then $lu = lv, mv = mw$ for some $l, m \in E(S)$. Now by the left invertive, paramedial, medial laws and the assumption, we have

$$\begin{aligned}(lm)u = (ll \cdot m)u = um \cdot ll &= lm \cdot lu = lm \cdot lv = \\ = ll \cdot mv = ll \cdot mw = wl \cdot ml &= wm \cdot ll = wm \cdot l = lm \cdot w.\end{aligned}$$

Thus $lm \cdot u = lm \cdot w$. Since $lm \in E(S)$, so $u\eta w$ equivalently η is transitive. Thus η is an equivalence relation. Now let $u\eta v, w \in S$. Then $lu = lv$ for some $l \in E(S)$, and

$$l(uw) = ll \cdot uw = lu \cdot lw = lv \cdot lw = ll \cdot vw = l(vw) \Rightarrow uw\eta vw.$$

Similarly, $wu\eta wv$. Thus η is compatible and hence is a congruence on S. Hence the result proved. □

Theorem 2.4. *Let S be a paramedial AG-groupoid. Then the relation μ defined on S with*

$$\mu = \{(u, v) \in S \times S : \quad xu = xv, \forall x \in S\}$$

is a congruence relation on S.

The following result holds for a more general class of inverse paramedial AG-groupoid.

Theorem 2.5. *Let S be an inverse AG-groupoid. Then the relation ρ defined on S with*

$$\rho = \left\{(u, v) \in S \times S : \quad u^{-1}u = v^{-1}v\right\} \tag{2. 3}$$

is a congruence relation on S.

Proof. Clearly, ρ is an equivalence relation. Now for left compatibility, let $u, v, w \in S$ such that $u\rho v$. Then we have

$$\begin{aligned}(wu)^{-1}(wu) &= (w^{-1}u^{-1})(wu) \\ &= (w^{-1}w)(u^{-1}u) \\ &= (w^{-1}w)(v^{-1}v) \\ &= (w^{-1}v^{-1})(wv) \\ &= (wv)^{-1}(wv) \\ \Rightarrow (wu)^{-1}(wu) &= (wv)^{-1}(wv) \Rightarrow u\rho v \Rightarrow wu\rho wv.\end{aligned}$$

Similarly, $uw\rho vw$. Hence ρ is compatible. Thus ρ is a congruence relation. □

Example 2.6. *Consider Example* (1.2)*, then the relation ρ defined in Equation* (2.3) *is given as under,*

$$\begin{aligned}\rho &= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4), \\ &\quad (3,1),(3,2),(3,3),(3,4)(4,1),(4,2),(4,3),(4,4)\}\end{aligned}$$

is a congruence relation.

Similarly, for Table 2 of an AG-groupoid $(S, \cdot)$ the ρ defined in Equation (2.3) as under is a congruence relation on S.

$$\rho = \{(1,1),\ (2,2),\ (3,3),\ (4,4),\ (5,5)\}$$

$\cdot$	1	2	3	4	5
1	1	3	2	5	4
2	4	5	3	1	2
3	5	2	4	3	1
4	3	4	1	2	5
5	2	1	5	4	3

Table 2

3. Natural partial order

Here we discuss natural partial relation on an inverse paramedial AG-groupoid S and investigate some of its properties. We start with the following theorem.

Theorem 3.1. *Let S be an inverse paramedial AG-groupoid. Then the relation $\leq$,*

$$u \leq v \Leftrightarrow u = uu^{-1} \cdot v \tag{3. 4}$$

is a partial order relation and is compatible on S.

Proof. The relation $\leq$ is clearly reflexive as S is inverse paramedial AG-groupoid.

$\leq$ is anti-symmetric: Assume that $u \leq v$ and $v \leq u$, then $u = uu^{-1} \cdot v$ and $v = vv^{-1} \cdot u$. Thus by assumption, left invertive, paramedial and medial laws,

$$\begin{aligned} u = uu^{-1} \cdot v &= ((uu^{-1} \cdot v)u^{-1})(vv^{-1} \cdot u) \\ &= (u^{-1}v \cdot uu^{-1})(vv^{-1} \cdot u) \\ &= (u^{-1}v \cdot vv^{-1})(uu^{-1} \cdot u) \\ &= (v^{-1}v \cdot vu^{-1})(uu^{-1} \cdot u) \\ &= (v^{-1}v \cdot uu^{-1})(vu^{-1} \cdot u) \\ &= (v^{-1}v \cdot uu^{-1})(uu^{-1} \cdot v) \\ &= (v^{-1}v \cdot uu^{-1})(u) \\ &= (u \cdot uu^{-1})(v^{-1}v) \\ &= (uv^{-1})(uu^{-1} \cdot v) \\ &= (vv^{-1})(uu^{-1} \cdot u) \\ &= vv^{-1} \cdot u \\ &= v. \end{aligned}$$

Thus $u = v$. Hence $\leq$ is anti-symmetric.

$\leq$ is transitive: To this end, assume that $u \leq v$ and $v \leq w$ this gives that

$$u = uu^{-1} \cdot v \tag{3. 5}$$

$$\text{and} \quad v = vv^{-1} \cdot w \tag{3. 6}$$

Now, using Equations (3.5) and (3.6), left invertive law, medial law, paramedial law and reflexive property, we have

$$\begin{aligned} u &= uu^{-1}\cdot v = (uu^{-1})(vv^{-1}\cdot w) \\ &= ((uu^{-1}\cdot v)u^{-1})(vv^{-1}\cdot w) \\ &= (u^{-1}v\cdot uu^{-1})(vv^{-1}\cdot w) \\ &= (u^{-1}v\cdot vv^{-1})(uu^{-1}\cdot w) \\ &= ((vv^{-1}\cdot v)u^{-1})(uu^{-1}\cdot w) \\ &= (vu^{-1})(uu^{-1}\cdot w) \\ &= (wu^{-1})(uu^{-1}\cdot v) \\ &= wu^{-1}\cdot u \\ &= uu^{-1}\cdot w \\ \Rightarrow u &\leq w. \end{aligned}$$

Equivalently, $\leq$ is transitive, and thus the relation $\leq$ is a partial order on S.
Next, for left compatibility, assume that $u \leq v$ and $w \in S$. Then

$$\begin{aligned} wu = w(uu^{-1}\cdot v) &= (ww^{-1}\cdot w)(uu^{-1}\cdot v) \\ &= (ww^{-1}\cdot uu^{-1})(wv) \\ &= (wu\cdot w^{-1}u^{-1})wv \\ &= (wu\cdot (wu)^{-1})wv \Rightarrow wu \leq wv. \end{aligned}$$

Hence the relation $\leq$ is left compatible. Further,

$$\begin{aligned} uw = (uu^{-1}\cdot v)w = (uu^{-1}\cdot v)(ww^{-1}\cdot w) & \\ &= (uu^{-1}\cdot ww^{-1})(vw) \\ &= (uw\cdot u^{-1}w^{-1})vw \\ &= (uw\cdot (uw)^{-1})vw \Rightarrow uw \leq vw. \end{aligned}$$

Thus the relation $\leq$ is right compatible, and whence is compatible. □

It is illustrated in the following that the relation $\leq$ defined on an inverse paramedial AG-grupoid is a compatible partial order.

Example 3.2. *See Example (1.2), the partial order $\leq$ as defined with (3. 4) and given below, is a compatible partial order on AG-groupoid S.*

$$\leq = \{(1,1),(2,2),(3,3),(4,4)\}$$

Corollary 3.3. *Let S be an inverse paramedial AG-groupoid and $u, v \in S$. Then $u \leq v \Leftrightarrow uu^{-1} = vu^{-1}$.*

Proof. Let $u \leq v$. Then

$$\begin{aligned} uu^{-1} &= (uu^{-1} \cdot v)u^{-1} \quad \text{(By Theorem 3.1)} \\ &= \left[(uu^{-1})(vv^{-1} \cdot v)\right] u^{-1} \quad \text{(By Theorem 3.1)} \\ &= \left[(v \cdot vv^{-1})(u^{-1}u)\right] u^{-1}) \quad \text{(By bi-commutative)} \\ &= (u^{-1} \cdot u^{-1}u)(v \cdot vv^{-1}) \quad \text{(By left invertive law)} \\ &= (vv^{-1} \cdot v)(u^{-1}u \cdot u^{-1}) \quad \text{((By bi-commutative))} \\ &= vu^{-1} \quad \text{(By Theorem 3.1).} \end{aligned}$$

Conversely, let $u, v \in S$. Then

$$uu^{-1} = vu^{-1} \Rightarrow uu^{-1} \cdot u = vu^{-1} \cdot u \Rightarrow u = uu^{-1} \cdot v.$$

Thus $u \leq v$. Hence the result is proved. □

In inverse AG-groupoid uu^{-1} and $u^{-1}u$ are not necessarily idempotent as shown in Table 3 and Table 4**.**

$*$	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

Table 3

$\cdot$	1	2	3	4
1	2	3	1	4
2	4	1	3	2
3	3	2	4	1
4	1	4	2	3

Table 4

In the above Table 3, $(S, *)$ is an inverse AG-groupoid such that the inverses of $1, 2, 3, 4$ are $4, 2, 3, 1$ respectively. Clearly, $(1 * 4) * (1 * 4) \neq 1 * 4$. Similarly, $(S, \cdot)$ in Table 4 is an inverse AG-groupoid such that the inverses of $1, 2, 3, 4$ are $4, 3, 2, 1$ respectively. Clearly, $(1 \cdot 2) \cdot (1 \cdot 2) \neq 1 \cdot 2$. However, in completely inverse AG-groupoid uu^{-1} and $u^{-1}u$ are idempotents, which is proved in Lemma 4.1.

4. Normal congruence pair

Let S denotes a completely inverse paramedial AG-grupoid in which in which we have $uu^{-1} = u^{-1}u$ or equivalently $uu^{-1}, u^{-1}u \in E(S)$ holds for each $u \in S$. Then the following lemma holds.

Lemma 4.1. *Let S be an inverse paramedial AG-groupoid, $u \in S$. Then*

$$uu^{-1}, u^{-1}u \in E(S) \Leftrightarrow uu^{-1} = u^{-1}u.$$

Proof. Let $uu^{-1} = u^{-1}u$. Then

$$(uu^{-1})^2 = uu^{-1} \cdot uu^{-1} = u^{-1}u \cdot uu^{-1} = (uu^{-1} \cdot u)u^{-1} = uu^{-1}.$$

Conversely, let $uu^{-1}, u^{-1}u \in E(S)$. Then

$$\begin{aligned}
uu^{-1} &= uu^{-1} \cdot uu^{-1} \\
&= (uu^{-1} \cdot uu^{-1})uu^{-1} \\
&= ((uu^{-1} \cdot u^{-1})u)uu^{-1} \\
&= (uu^{-1} \cdot u)(uu^{-1} \cdot u^{-1}) \\
&= (u^{-1}u)(uu^{-1} \cdot uu^{-1}) \\
&= (u^{-1}u)((uu^{-1} \cdot u^{-1})u) \\
&= (u^{-1}u)((uu^{-1} \cdot u^{-1})(uu^{-1} \cdot u)) \\
&= (u^{-1}u)((uu^{-1} \cdot uu^{-1})(u^{-1}u)) \\
&= (u^{-1}u)(uu^{-1} \cdot u^{-1}u) \\
&= (u^{-1}u)((u^{-1}u \cdot u^{-1})u) \\
&= (u^{-1}u)(u^{-1}u) \\
&= u^{-1}u.
\end{aligned}$$

Hence the lemma is proved. □

The following is a consequence of Theorem (3.1)

Corollary 4.2. *Let S be a completely inverse paramedial AG-groupoid and $u, v \in S$. Then $u \leq v \Leftrightarrow (\exists g \in E(S))\, u = gv$.*

Proof. Let $u, v \in S$. Then $u \leq v$ if and only if, $u = uu^{-1} \cdot v$. Since $uu^{-1} \in E(S)$, therefore if $g = uu^{-1}$ then $u = gv$.
Conversely, let $u, v \in S$ be such that $g \in E(S)$ and $u = gv$. Since $uu^{-1} = u^{-1}u \in E(S)$ and $E(S)$ is a semi-lattice, we have

$$\begin{aligned}
uu^{-1} \cdot v &= (gv \cdot gv^{-1})v \\
&= (gv \cdot gv^{-1})(vv^{-1} \cdot v) \\
&= (gv \cdot vv^{-1})(gv^{-1} \cdot v) \\
&= ((vv^{-1} \cdot v)g)(vv^{-1} \cdot g) \\
&= (vg)(vv^{-1} \cdot g) \\
&= (gg)(vv^{-1} \cdot v) \\
&= gv \Rightarrow uu^{-1} \cdot v \\
&= u.
\end{aligned}$$

Thus for each $u, v, x, y \in S$, we have $xu \leq xv \Rightarrow ux \leq vx$ and so $u \leq v$. □

Lemma 4.3. *Let $(S, \cdot, \leq)$ be a separative order inverse paramedial AG-groupoid. Then for each $u, v, x, y \in S$ we have,*

(1) $xu \leq xv \Leftrightarrow ux \leq vx$,
(2) $x^2u \leq x^2v \Leftrightarrow xu \leq xv$.

Proof. Let $u, v \in S$. Then

(1) $xu \leq xv$. Since $xu \cdot xu \leq xv \cdot xu$, we have

$$\begin{aligned} xu \leq xv &\Rightarrow xu \cdot xu \leq xv \cdot xu \\ &\Rightarrow uu \cdot xx \leq uv \cdot xx \\ &\Rightarrow ux \cdot ux \leq uv \cdot xx \\ &\Rightarrow (ux)^2 \leq ux \cdot vx. \end{aligned} \tag{4. 7}$$

Also

$$\begin{aligned} xu \leq xv &\Rightarrow xu \cdot xv \leq xv \cdot xv \\ &\Rightarrow vu \cdot xx \leq vv \cdot xx \\ &\Rightarrow vx \cdot ux \leq vx \cdot vx \\ &\Rightarrow vx \cdot ux \leq (vx)^2 \end{aligned} \tag{4. 8}$$

Similarly, $xu \leq xv \Rightarrow (ux)^2 \leq vx \cdot ux$ and $ux \cdot vx \leq (vx)^2$. Hence $ux \leq vx$.

(2) $x^2u \leq x^2v$. Since $x^2u \cdot u \leq x^2v \cdot u$, we have

$$\begin{aligned} x^2u \leq x^2v \Rightarrow x^2u \cdot u \leq x^2v \cdot u & \\ &\Rightarrow uu \cdot x^2 \leq uv \cdot x^2 \\ &\Rightarrow ux \cdot ux \leq ux \cdot vx \\ &\Rightarrow (ux)^2 \leq ux \cdot vx. \end{aligned} \tag{4. 9}$$

Also

$$\begin{aligned} x^2u \leq x^2v \Rightarrow x^2u \cdot v \leq x^2v \cdot v & \\ &\Rightarrow vu \cdot x^2 \leq vv \cdot x^2 \\ &\Rightarrow vx \cdot ux \leq vx \cdot vx \\ &\Rightarrow vx \cdot ux \leq (vx)^2 \end{aligned} \tag{4. 10}$$

Similarly, $x^2u \leq x^2v \Rightarrow (ux)^2 \leq vx \cdot ux$ and $ux \cdot vx \leq (vx)^2$. Hence $ux \leq vx$ and by Part (1) $xu \leq xv$.

□

The following definitions are introduced in [3].

Definition 4.4. *Let K be a subset of a completely inverse AG-groupoid S. Then*

(1) *K is **full**, if $E(S) \subseteq K$;*
(2) *K is **self-conjugate**, if $u^{-1}(Ku) \subseteq K$, for every $u \in K$;*
(3) *K is **inverse closed**, if $u \in K \Rightarrow u^{-1} \in K$;*
(4) *K is **normal**, if K is full, self-conjugate and inverse closed;*
(5) *Let ρ be the congruence relation on S as defined in Theorem (2.5). Then restriction $\rho\mid_{E(S)}$ is the **trace of** ρ to be denoted by **tr** ρ;*
(6) *The set $ker\rho = \{u \in S \mid (\exists g \in E(S))\, u\rho g\}$.*

Example 4.5. *Let $S = \{w, x, y, z\}$. Then $(S, *)$ with the Table 5 is an inverse paramedial AG-groupoid such that each element is its own inverse. Clearly, $K = \{w, x\}$ is normal in S.*

*	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	z	y	w	x
z	y	z	x	w

Table 5

Lemma 4.6. *Let ρ be a congruence relation on S. Then $ker\rho$ is a normal subgroupoid of S.*

Proof. Since ρ is a congruence relation on S, so for any $u, v \in ker\rho$ there exists $l, m \in E(S)$ such that $u\rho l, v\rho m$. Now $uv\rho lm$, clearly $lm \in E(S)$. So $uv \in ker\rho$, hence $ker\rho$ is a subgroupoid of S. Clearly, $ker\rho$ is full. Now, let $u \in S$. Then $u^{-1}(ker\rho \cdot u) = \{u^{-1}(vu) \mid v \in ker\rho\}$. Since $v \in ker\rho$ so there exists $m \in E(S)$ such that $v\rho m$ so, $u^{-1}(vu)\rho u^{-1}(mu)$. Thus

$$\begin{aligned} u^{-1}(mu) &= (u^{-1}u \cdot u^{-1})(mu) \\ &= (uu^{-1})(m \cdot u^{-1}u) \\ &= (um)(u^{-1} \cdot uu^{-1}) \\ &= ((u^{-1}u) \cdot m)(u^{-1}u) \\ &= (m \cdot u^{-1}u)(u^{-1}u) \\ &= (u^{-1}u \cdot u^{-1}u)m \\ &= u^{-1}u \cdot m. \end{aligned}$$

Since $u^{-1}u \cdot m \in E(S)$ so $u^{-1}(vu) \in ker\rho$. Hence $u^{-1}(ker\rho u) \subseteq ker\rho$, and thus $ker\rho$ is self-conjugate subgroupoid of S. Also if $u \in ker\rho$ then $u\rho m$ for some $m \in E(S)$ and $u^{-1}\rho m^{-1} = m$. Hence $u^{-1} \in ker\rho$, and $ker\rho$ is inverse closed. Thus $ker\rho$ is normal subgroupoid of S. □

Definition 4.7. [8] *Let K be a normal subgroupoid of S and τ be a congruence on semilattice $E(S)$ such that,*

$$lu \in K, l\tau u^{-1}u \Rightarrow u \in K, \tag{4. 11}$$

for every $u \in S$ and $l \in E(S)$. Then the pair (K, τ) is a congruence pair for S.
In this case, we define a relation $\rho_{(K,\tau)}$ on S by

$$u\rho(K, \tau)v \Leftrightarrow u^{-1}u\tau v^{-1}v, uv^{-1}, vu^{-1} \in K.$$

Lemma 4.8. *For a congruence pair (K, τ) for S, we have*

$$l(uv) \in K, l\tau u^{-1}u \Rightarrow uv \in K$$

for any $u, v \in S, l \in E(S)$.

Proof. Let $u, v \in S, l \in E(S), l(uv) \in K$ and $l\tau u^{-1}u$. Then using the paramedial, medial, left invertive laws and definition of inverse AG-groupoid

$$\begin{aligned} l \cdot uv &= ll \cdot uv \\ &= vl \cdot ul \\ &= ((vv^{-1} \cdot v)l)(ul) \\ &= (lv \cdot vv^{-1})(ul) \\ &= (ul \cdot vv^{-1})(lv) \\ &= (ul \cdot l)(vv^{-1} \cdot v) \\ &= (ll \cdot u)(vv^{-1} \cdot v) \\ &= (lu)(vv^{-1} \cdot v) \\ &= (l \cdot vv^{-1})(uv) \end{aligned}$$

and

$$\begin{aligned} (uv)^{-1}(uv) &= (u^{-1}v^{-1})(uv) \\ &= (u^{-1}u)(v^{-1}v)\tau l(v^{-1}v). \end{aligned}$$

By above and (4. 11), we have $uv \in K$. □

Definition 4.9. [4] *Let K be a full subgroupoid of S and τ a congruence on $E(S)$ and $\leq$ be the relation as defined in Theorem (3.1) and satisfying the following condition:*

(1) *For all $u \in S, v \in K, v \leq u$ and $uu^{-1}\tau vv^{-1}$ imply $u \in K$.*
We call (K, τ) a pseudo normal congruence pair for S. If, in addition,
(2) *For every $u \in K$, there exists $v \in S$ with $v \leq u, uu^{-1}\tau vv^{-1}$ and $v^{-1} \in K$, then (K, τ) is called normal congruence pair for S.*

For pseudo normal congruence pair (K, τ), we define a relation, $\rho_{(K,\tau)}$ as follows:

$$u\rho_{(K,\tau)}v \Leftrightarrow uv^{-1}, u^{-1}v, vu^{-1}, v^{-1}u \in K, uu^{-1} \cdot vv^{-1}\tau uu^{-1}\tau vv^{-1}. \qquad (4.\ 12)$$

Lemma 4.10. *Let (K, τ) be a pseudo normal congruence pair of S, $u, v \in S$. If $u\rho_{(K,\tau)}v$ and $v \in K$, then $u \in K$.*

Proof. Since $u\rho(K, \tau)v$, so we have $uv^{-1} \in K$ and $uu^{-1} \cdot vv^{-1}\tau vv^{-1}$. Since $v \in K$ and K is full subgroupoid, so $uv^{-1} \cdot v = vv^{-1} \cdot u \in K$. We have to prove that $uv^{-1} \cdot v \leq u$.

Here

$$\begin{aligned}((uv^{-1}\cdot v)(uv^{-1}\cdot v)^{-1})u &= ((uv^{-1}\cdot v)(u^{-1}v\cdot v^{-1}))u\\ &= ((vv^{-1}\cdot u)(v^{-1}v\cdot u^{-1}))u\\ &= ((vv^{-1}\cdot v^{-1}v)(uu^{-1}))u\\ &= (vv^{-1}\cdot uu^{-1})u\\ &= (uu^{-1}\cdot vv^{-1})u\\ &= (uu^{-1}\cdot vv^{-1})(uu^{-1}\cdot u)\\ &= (uu^{-1}\cdot uu^{-1})(vv^{-1}\cdot u)\\ &= uu^{-1}(vv^{-1}\cdot u)\\ &= uu^{-1}(uv^{-1}\cdot v)\\ &= vu^{-1}(uv^{-1}\cdot u)\\ &= (v\cdot uv^{-1})(u^{-1}u)\\ &= (u^{-1}u\cdot uv^{-1})v\\ &= (v^{-1}u\cdot uu^{-1})v\\ &= ((uu^{-1}\cdot u)v^{-1})v\\ &= uv^{-1}\cdot v.\end{aligned}$$

Hence, by (3. 4), it follows that $uv^{-1}\cdot v\leq u$.
Also

$$\begin{aligned}(uv^{-1}\cdot v)(uv^{-1}\cdot v)^{-1} &= (uv^{-1}\cdot v)(u^{-1}v\cdot v^{-1})\\ &= (uv^{-1}\cdot u^{-1}v)vv^{-1}\\ &= (uu^{-1}\cdot v^{-1}v)vv^{-1}\\ &= (vv^{-1}\cdot v^{-1}v)\cdot uu^{-1}\\ &= vv^{-1}\cdot uu^{-1}\tau uu^{-1}.\end{aligned}$$

Hence by Definition (4.9(i)) it follows that $u\in K$. □

Theorem 4.11. *Let* (K,τ) *be a pseudo normal congruence pair for* S*. Then* $\rho_{(K,\tau)}$ *is a congruence on* S *with*

$$ker\rho_{(K,\tau)}=\left\{u\in K\mid(\exists v\in S),v\leq u,uu^{-1}\tau\,vv^{-1},v^{-1}\in K\right\}\tag{4. 13}$$

Proof. Let $\rho_{(K,\tau)}$, be a pseudo normal congruence pair for S as given in (4. 12) and $\rho=\rho_{(K,\tau)}$. First we show that ρ is compatible, for this assume $u\rho v$ and $w\in S$. Then

$$uw\cdot(vw)^{-1}=uw\cdot v^{-1}w^{-1}=uv^{-1}\cdot ww^{-1}\subseteq K\cdot E(S)\subseteq K,$$

By Definition (4.9), for *pseudo normal congruence pair* and K is full. So, $uw\cdot(vw)^{-1}\in K$. Similarly, $(vw)^{-1}\cdot uw,(uw)^{-1}\cdot vw,vw\cdot(uw)^{-1}\in K$.

Next,

$$\begin{aligned}
(uw\cdot(uw)^{-1})((vw)^{-1}\cdot vw) &= (uw\cdot(vw)^{-1})((uw)^{-1}\cdot vw)\\
&= (uw\cdot v^{-1}w^{-1})(u^{-1}w^{-1}\cdot vw)\\
&= (uv^{-1}\cdot ww^{-1})(u^{-1}v\cdot w^{-1}w)\\
&= (uv^{-1}\cdot u^{-1}v)(ww^{-1}\cdot w^{-1}w)\\
&= (uv^{-1}\cdot u^{-1}v)(ww^{-1}\cdot ww^{-1})\\
&= (uu^{-1}\cdot v^{-1}v)(ww^{-1}\cdot ww^{-1})\\
&= (uu^{-1}\cdot v^{-1}v)ww^{-1}\tau uu^{-1}\cdot ww^{-1}\\
(uw\cdot(uw)^{-1})((vw)^{-1}\cdot vw) &\;\tau\; (uw\cdot(uw)^{-1}).
\end{aligned}$$

By symmetry, it follows that

$$(uw\cdot(uw)^{-1})((vw)^{-1}\cdot vw)\quad\tau\quad(vw\cdot(vw)^{-1}).$$

Hence $uw\rho vw$. Thus ρ is right compatible, similarly, ρ is left compatible, thus ρ is compatible.

Now, we have to show that ρ is an equivalence. Since K is full, so ρ is reflexive. Obviously, ρ is symmetric. For transitivity, let $u\rho v, v\rho w$. Then by right compatibility $uw^{-1}\rho vw^{-1}$ and $vw^{-1}\rho ww^{-1}$, since $ww^{-1}\in E(S)\subseteq K$, and $vw^{-1}\rho ww^{-1}$, so $vw^{-1}\in K$ (by Lemma (4.10)). Again $uw^{-1}\rho vw^{-1}$ so again by Lemma (4.10), $uw^{-1}\in K$. Similarly, $uu^{-1}\rho vu^{-1}, vu^{-1}\rho wu^{-1}\Rightarrow wu^{-1}\in K$ (by Lemma (4.10)).

Similarly, by left compatibility $u\rho v, v\rho w$ implies $u^{-1}u\rho u^{-1}v$ and $u^{-1}v\rho u^{-1}w$, and $w^{-1}v\rho w^{-1}w$ so again by Lemma (4.10), we have $u^{-1}w, w^{-1}u\in K$.

Also $u\rho v, v\rho w$ yields

$$u^{-1}u\cdot vv^{-1}\tau uu^{-1}\tau vv^{-1}, v^{-1}v\cdot ww^{-1}\tau vv^{-1}\tau ww^{-1}.$$

and by transitivity it follows that $uu^{-1}\tau ww^{-1}$. Moreover,

$$\begin{aligned}
(vv^{-1}\cdot ww^{-1})(uu^{-1}\cdot ww^{-1}) &= (vv^{-1}\cdot uu^{-1})(ww^{-1}\cdot ww^{-1})\\
&= (vv^{-1}\cdot uu^{-1})ww^{-1}\tau uu^{-1}\cdot ww^{-1},
\end{aligned}$$

also

$$\begin{aligned}
(vv^{-1}\cdot ww^{-1})(uu^{-1}\cdot ww^{-1}) &= (vv^{-1}\cdot uu^{-1})(ww^{-1}\cdot ww^{-1})\\
&= (vv^{-1}\cdot uu^{-1})ww^{-1}\tau vv^{-1}\cdot ww^{-1}\tau ww^{-1}.
\end{aligned}$$

Whence, $uu^{-1}\cdot ww^{-1}\tau ww^{-1}$.

Now, $uw^{-1}, u^{-1}w, wu^{-1}, w^{-1}u\in K, uu^{-1}\cdot ww^{-1}\tau uu^{-1}\tau ww^{-1}$ is equivalent to $u\rho w$. Hence ρ is transitive relation and so is a congruence. □

5. Conclusions

In this article, the concept of inverse AG-groupoid [4, 7] is extended to paramedial AG-groupoid S that satisfies the paramedial law: $uv\cdot wx = xv\cdot wu$, and various of its properties are investigated. It is proved that inverses in an inverse paramedial AG-groupoid are unique. Congruences, partial order, and compatible partial orders for inverse paramedial AG-groupoid are introduced and investigated. This idea is further proceeded to completely inverse paramedial AG-groupoids. Various notions for completely inverse paramedial AG-groupoids are defined and investigated. Furthermore, some congruences on completely inverse paramedial AG-groupoids are introduced and characterized. The concept of separative ordered and completely separative, normal sub-groupoid, pseudo normal congruence pair, and normal congruence pair for the class of completely inverse paramedial AG-groupoids are also introduced and investigated. Various examples are provided for justification of the produced results.

AUTHOR CONTRIBUTIONS

M. Rashad and I. Ahmad developed the theoretical formalism, gave the examples. F. Karaaslan contributed to the final version of the manuscript.

ACKNOWLEDGMENT

This work is financially supported by the Higher Education Commission, Pakistan (HEC) under the project NRPU-3509.

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