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Solution of nonlinear equations using three point Gaussian quadrature formula and decomposition technique

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Abstract.: The problem of solving nonlinear equations (real or complex) is a nontrivial task in many areas of science and engineering. Usually, the analytic methods for such equations are not directly affordable and require an iterative approach for getting an approximate solution. Keeping in view the above facts, we suggest and analyze some new iterative methods for solving nonlinear equation of the form $f(u) = 0$ by using the decomposition technique coupled with a system of equations and three-points Gaussian quadrature formula. We also determine the convergence order of our proposed iterative methods. Some test examples are given to endorse and validate the performance of new methods as compared to previously well-known methods.

Key Words: Convergence analysis, Gaussian three point quadrature formula, Daftardar-Gejji decomposition technique, Nonlinear equation, Iterative methods.

1. Introduction

It is widely known that solving nonlinear equation $f(u) = 0$ is one of the key considerable problem in Mathematical sciences particularly in numerical analysis. Many researchers have proposed, analyzed, and modified diversity of numerical methods using different techniques such as: Taylor's series, modified homotopy perturbation method, decomposition method, variational iterative method, and quadrature formulas for numerous enhancements. A vast literature is available that highlight different methods for the solution of nonlinear equations for example, see [1, 5, 7, 8, 10–16, 19–24, 29, 34, 35] and references therein. Abbasbandy [1] has applied ADM (Adomian decomposition method) to present some new modifications in the Newton-Raphson method. Later on, Chun [7] has constructed a sequence of multistep higher-order iterative methods by using the same technique. Implementation of ADM requires the involvement of higher-order polynomial and its derivative that is itself a problem. To overcome this weakness of ADM several new techniques have been suggested and analyzed. Another simple decomposition technique was introduced by Daftardar-Gejji and Jafari [9] that does not require higher-order derivative evaluation. The purpose of this technique was to solve a variety of diverse problems and plays a central role in suggesting iterative methods for solving nonlinear equations. Noor et al. [25, 26] have used this technique and derived various order families of convergent iterative methods by writing the nonlinear function in a new series representation with the help of quadrature formula and the fundamental theorem of calculus. Saqib and Iqbal [30] have constructed the sequence of multistep iterative methods using this technique and derived a comparative study of numerical results. Alharbi et al. [2] have introduced some new and efficient iterative methods and implemented a decomposition technique along with an auxiliary function. Quadrature rules are another effective and veritable tools that are employed to develop effective iterative methods for getting approximate, converging solutions of the nonlinear equations. Quadrature-based iterative methods are developed by approximating the indefinite integral representation of Newton's method to Taylor's expansion of vector function and employing quadrature formulas. This domain is adressed by considering classical quadrature formulas see [17, 34]. Furthermore, the connection has been re-established by Noor [28] and derived the quadrature-based fifth-order convergent method by using the specially derived Gaussian quadrature formula for solving non-linear equations. By implementing the decomposition technique [9], Ogbereyivwe and Muka [32] have developed quadrature-based families of iterative methods for solving system of nonlinear equations subject to singular jacobian. Ali et al. [3, 4] have generalized and developed a family of iterative methods. They testified the performance of these methods by numerical and graphical analysis of some nonlinear equations. Based on the Gauss quadrature rule Srivastava et al. [31] have investigated sixth-order three-step Newtons' method for solving a system of nonlinear equations. These methods are also helpful to evaluate nonlinear boundary-value problems and integral equations in fewer iterations with higher accuracy.

Motivated by the research going on in this direction. In this article, we first develop a class of quadrature-based multistep iterative methods by applying decomposition technique and fundamental law of calculus along with the three-point Gaussian quadrature formula [6]. These quadrature-based methods are of implict type, in which we use Newton's method as predictor and iterative function generated from quadrature rule as corrector; see Section 2. Section 3 deals with the convergence analysis and calculations of error equations of these algorithms. Some numerical real-world examples of nonlinear equations are considered in Section 4 to check the efficiency of the predictor-corrector type schemes. Moreover, a comparative analysis of these methods with some other methods is presented to indicate an efficient refinement $\&$ alternative of already developed convergent methods of the same order. Finally, as a part of our concluding remarks, we discuss the accumulated facts of our findings.

2. NEW CLASS OF FAMILY OF ITERATIVE SCHEMES

Consider the nonlinear equation

$$f(u) = 0, \qquad u \in \mathbb{R}. \tag{2. 1}$$

Where β is an initial guess suffitiently close to α which is simple root of equation (2. 1). Now, utilizing the technique of Noor et al. [25, 26] approximate the function $f(u)$ using fundamental theorem of calculus and quadrature formula:

$$f(u) = f(\beta)+\frac{u-\beta}{18}\left\{5f' \ \ \frac{u+\beta}{2}-\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ +8f' \ \ \frac{u+\beta}{2} \ \ +5f' \ \ \frac{u+\beta}{2}+\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ \right\}. \tag{2. 2}$$

Applying the technique of He [18] and writing the nonlinear equation as an equivalent coupled system of equations:

$$f(\beta)+\frac{u-\beta}{18}\left\{5f' \ \ \frac{u+\beta}{2}-\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ +8f' \ \ \frac{u+\beta}{2} \ \ +5f' \ \ \frac{u+\beta}{2}+\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ \right\}+g(u), \tag{2. 3}$$

$$g(u) = f(u)-f(\beta)-\frac{u-\beta}{18}\left\{5f' \ \ \frac{u+\beta}{2}-\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ +8f' \ \ \frac{u+\beta}{2} \ \ +5f' \ \ \frac{u+\beta}{2}+\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ \right\}. \tag{2. 4}$$

It can be rephrased as

$$u = \beta-\frac{18 \ \ f(\beta)+g(u)}{\left\{5f' \ \ \frac{u+\beta}{2}-\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ +8f' \ \ \frac{u+\beta}{2} \ \ +5f' \ \ \frac{u+\beta}{2}+\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ \right\}}, \tag{2. 5}$$

$$u = c+\mathcal{N}(u). \tag{2. 6}$$

where

$$c = \beta, \tag{2. 7}$$

and
$$\mathcal{N}(u) = -\frac{18 \ \ f(\beta)+g(u)}{\left\{5f' \ \ \frac{u+\beta}{2}-\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ +8f' \ \ \frac{u+\beta}{2} \ \ +5f' \ \ \frac{u+\beta}{2}+\frac{u-\beta}{2}\sqrt{\frac{3}{5}} \ \ \right\}}. \tag{2. 8}$$

It is clear that $\mathcal{N}(u)$ is a nonlinear operator. Now we construct sequence of higher order iterative methods by employing decomposition technique initiated by Daftardar-Gejji and Jafari [9]. With the support of this technique solution of Eq. (2. 1) can be represented as in terms of the infinite series:

$$u = \sum_{r=0}^{\infty} u_r. \tag{2. 9}$$

The nonlinear operator can be decomposed as:

$$\mathcal{N}(u) = \mathcal{N}(u_0)+\sum_{r=1}^{\infty}\left\{\mathcal{N}\left(\sum_{q=0}^{r} u_q\right)-\mathcal{N}\left(\sum_{q=0}^{r-1} u_q\right)\right)\right\}. \tag{2. 10}$$

Thus from equation (2. 6), (2. 9), (2. 10), we have

$$\sum_{r=0}^{\infty} u_r = c+\mathcal{N}(u_0)+\sum_{r=1}^{\infty}\left\{\mathcal{N}\left(\sum_{q=0}^{r} u_q\right)-\mathcal{N}\left(\sum_{q=0}^{r-1} u_q\right)\right\}. \tag{2. 11}$$

which generates the following iterative scheme:

$$\begin{cases} u_0 &= c, \\ u_1 &= \mathcal{N}(u_0), \\ u_2 &= \mathcal{N}(u_0+u_1)-\mathcal{N}(u_0), \\ & \vdots \\ u_{m+1} &= \mathcal{N}\left(\sum_{q=0}^{m} u_q\right)-\mathcal{N}\left(\sum_{q=0}^{m-1} u_q\right), \qquad m=1,2,\ldots. \end{cases} \tag{2. 12}$$

It follows that

$$u_1+u_2+\cdots+u_{m+1}=\mathcal{N}\big(u_0+u_1+u_2+\cdots+u_m\big), \qquad m=1,2,\ldots \tag{2. 13}$$

and

$$u=c+\sum_{r=1}^{\infty} u_r. \tag{2. 14}$$

Notable approximation of u is conveyed as:

$$\lim_{m\to\infty} U_m = u, \tag{2. 15}$$

$$\text{and} \qquad U_m = u_0+u_1+\cdots u_m. \tag{2. 16}$$

For m **= 0**, we have

$$\mathbf{u} \approx \mathbf{U_0} = \mathbf{u_0} = \mathbf{c} = \beta$$

It follows from (2. 4) and (2. 7), we get

$$g(u_0)=0. \tag{2. 17}$$

Considering (2. 8), (2. 12) and (2. 17), we obtain:

$$\begin{aligned} u_1=\mathcal{N}(u_0) &= -\frac{18\big(f(\beta)+g(u_0)\big)}{5f'\Big(\frac{u_0+\beta}{2}-\frac{u_0-\beta}{2}\sqrt{\frac{3}{5}}\Big)+8f'\Big(\frac{u_0+\beta}{2}\Big)+5f'\Big(\frac{u_0+\beta}{2}+\frac{u_0-\beta}{2}\sqrt{\frac{3}{5}}\Big)}, \\ &= -\frac{18\big(f(\beta)\big)}{5f'\Big(\frac{u_0+\beta}{2}-\frac{u_0-\beta}{2}\sqrt{\frac{3}{5}}\Big)+8f'\Big(\frac{u_0+\beta}{2}\Big)+5f'\Big(\frac{u_0+\beta}{2}+\frac{u_0-\beta}{2}\sqrt{\frac{3}{5}}\Big)}. \end{aligned} \tag{2. 18}$$

For $m=1$, we have

$$\mathbf{u} \approx \mathbf{U_1} = \mathbf{u_0}+\mathbf{u_1} = \mathbf{u_0}+\mathcal{N}(\mathbf{u_0})$$

$$\begin{aligned} u &= \beta-\frac{18f(\beta)}{5f'\Big(\frac{u+\beta}{2}-\frac{u-\beta}{2}\sqrt{\frac{3}{5}}\Big)+8f'\Big(\frac{u+\beta}{2}\Big)+5f'\Big(\frac{u+\beta}{2}+\frac{u-\beta}{2}\sqrt{\frac{3}{5}}\Big)}, \\ &= u_0-\frac{f(u_0)}{f'(u_0)}. \end{aligned} \tag{2. 19}$$

This formulation suggests the following algorithm:

Algorithm 2(a): For a given initial guess u_0, approximate solution u_{n+1} is determined by the following iterative scheme:

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)}, \qquad f'(u_n) \neq 1, \qquad \text{n} = 0, 1, 2, \ldots \tag{2. 20}$$

This is famous Newton's method which have 2nd order convergence [27].

It is noted that

$$u_0 + u_1 - \beta = -\frac{f(\beta)}{f'(\beta)}, \tag{2. 21}$$

From (2. 4), (2. 8) and (2. 21), we have

$$g \;\; u_0 + u_1 \;\; = f(u_0 + u_1) - f(\beta) - \frac{f(\beta)}{18f'(\beta)}\Big\{5f' \;\; \frac{u_0 + u_1 + \beta}{2} - \frac{u_0 + u_1 - \beta}{2}\sqrt{\frac{3}{5}} \;\; + 8f' \;\; \frac{u_0 + u_1 + \beta}{2} \;\; + 5f' \;\; \frac{u_0 + u_1 + \beta}{2} + \frac{u_0 + u_1 - \beta}{2}\sqrt{\frac{3}{5}} \;\; \Big\}, \tag{2. 22}$$

$$u_1 + u_2 = \mathcal{N}(u_0 + u_1) = -\frac{f(\beta)}{f'(\beta)} - \frac{18f(u_0 + u_1)}{5f' \;\; \frac{u_0+u_1+\beta}{2} - \frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}} \;\; + 8f' \;\; \frac{u_0+u_1+\beta}{2} \;\; + 5f' \;\; \frac{u_0+u_1+\beta}{2} + \frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}}.$$

For $m = 2$, we have

$$\mathbf{u} \approx \mathbf{U_2} = \mathbf{u_0} + \mathbf{u_1} + \mathbf{u_2} = \mathbf{c} + \mathcal{N}\left(\mathbf{u_0} + \mathbf{u_1}\right)$$

$$= \beta - \frac{f(\beta)}{f'(\beta)} - \frac{18f(u_0 + u_1)}{5f' \;\; \frac{u_0+u_1+\beta}{2} - \frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}} \;\; + 8f' \;\; \frac{u_0+u_1+\beta}{2} \;\; + 5f' \;\; \frac{u_0+u_1+\beta}{2} + \frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}}, \tag{2. 23}$$

This formulation yields the following two step method for solving nonlinear equation:

Algorithm 2(b): For a given x_0, approximate solution x_{n+1} is computed by the following iterative scheme:

$$y_n = u_n - \frac{f(u_n)}{f'(u_n)}, \tag{2. 24}$$

$$u_{n+1} = y_n - \frac{18f(y_n)}{5f' \;\; \frac{y_n+u_n}{2} - \frac{y_n-u_n}{2}\sqrt{\frac{3}{5}} \;\; + 8f' \;\; \frac{y_n+u_n}{2} \;\; + 5f' \;\; \frac{y_n+u_n}{2} + \frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}} \qquad \text{n} = 0, 1, 2, \ldots . \tag{2. 25}$$

It is noted that

$$u_0 + u_1 + u_2 = \beta - \frac{f(\beta)}{f'(\beta)} - \frac{18f(u_0 + u_1)}{5f' \;\; \frac{u_0+u_1+\beta}{2} - \frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}} \;\; + 8f' \;\; \frac{u_0+u_1+\beta}{2} \;\; + 5f' \;\; \frac{u_0+u_1+\beta}{2} + \frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}},$$

$$g\left(u_0+u_1+u_2\right) = f(u_0+u_1+u_2)-f(\beta)-\frac{f(\beta)}{18f'(\beta)}\Big\{5f'\left(\frac{u_0+u_1+u_2+\beta}{2}-\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)$$
$$+8f'\left(\frac{u_0+u_1+u_2+\beta}{2}\right)+5f'\left(\frac{u_0+u_1+u_2+\beta}{2}+\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)\Big\},$$

$$u_1+u_2+u_3 = N(u_0+u_1+u_2) = \beta-\frac{f(\beta)}{f'(\beta)}$$
$$-\frac{18f(u_0+u_1)}{5f'\left(\frac{u_0+u_1+\beta}{2}-\frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{u_0+u_1+\beta}{2}\right)+5f'\left(\frac{u_0+u_1+\beta}{2}+\frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}\right)}$$
$$-\frac{18f(u_0+u_1+u_2)}{5f'\left(\frac{u_0+u_1+u_2+\beta}{2}-\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{u_0+u_1+u_2+\beta}{2}\right)+5f'\left(\frac{u_0+u_1+u_2+\beta}{2}+\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)}. \quad (2.\ 26)$$

For $m = 3$

$$\mathbf{u} \approx \mathbf{U_3} = \mathbf{u_0}+\mathbf{u_1}+\mathbf{u_2}+\mathbf{u_3} = \mathbf{c}+\mathcal{N}\left(\mathbf{u_0}+\mathbf{u_1}+\mathbf{u_2}\right)$$

$$= \beta-\frac{f(\beta)}{f'(\beta)}-\frac{18f(u_0+u_1)}{5f'\left(\frac{u_0+u_1+\beta}{2}-\frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{u_0+u_1+\beta}{2}\right)+5f'\left(\frac{u_0+u_1+\beta}{2}+\frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}\right)}$$
$$-\frac{18f(u_0+u_1+u_2)}{5f'\left(\frac{u_0+u_1+u_2+\beta}{2}-\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{u_0+u_1+u_2+\beta}{2}\right)+5f'\left(\frac{u_0+u_1+u_2+\beta}{2}+\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)},$$

This formulation allows us to suggest the following Algorithm:

Algorithm 2(c): For a given u_0, approximate solution u_{n+1} is computed by the following iterative scheme:

$$y_n = u_n-\frac{f(u_n)}{f'(u_n)}, \quad (2.\ 27)$$

$$z_n = y_n-\frac{18f(y_n)}{5f'\left(\frac{y_n+u_n}{2}-\frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{y_n+u_n}{2}\right)+5f'\left(\frac{y_n+u_n}{2}+\frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\right)}, \quad (2.\ 28)$$

$$u_{n+1} = z_n-\frac{18f(z_n)}{5f'\left(\frac{z_n+u_n}{2}-\frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{z_n+u_n}{2}\right)+5f'\left(\frac{z_n+u_n}{2}+\frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\right)}, \quad \text{n} = 0,1,2,\ldots. \quad (2.\ 29)$$

For $m = 4$, we have

$$\mathbf{u} \approx \mathbf{U_4} = \mathbf{u_0}+\mathbf{u_1}+\mathbf{u_2}+\mathbf{u_3}+\mathbf{u_4} = \mathbf{c}+\mathcal{N}\left(\mathbf{u_0}+\mathbf{u_1}+\mathbf{u_2}+\mathbf{u_3}\right)$$

$$=\beta-\frac{f(\beta)}{f'(\beta)}-\frac{18f(u_0+u_1)}{5f'\left(\frac{u_0+u_1+\beta}{2}-\frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{u_0+u_1+\beta}{2}\right)+5f'\left(\frac{u_0+u_1+\beta}{2}+\frac{u_0+u_1-\beta}{2}\sqrt{\frac{3}{5}}\right)}$$
$$-\frac{18f(u_0+u_1+u_2)}{5f'\left(\frac{u_0+u_1+u_2+\beta}{2}-\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{u_0+u_1+u_2+\beta}{2}\right)+5f'\left(\frac{u_0+u_1+u_2+\beta}{2}+\frac{u_0+u_1+u_2-\beta}{2}\sqrt{\frac{3}{5}}\right)}$$
$$-\frac{18f(\tilde{u})}{5f'\left(\frac{\tilde{u}+\beta}{2}-\frac{\tilde{u}-\beta}{2}\sqrt{\frac{3}{5}}\right)+8f'\left(\frac{\tilde{u}+\beta}{2}\right)+5f'\left(\frac{\tilde{u}+\beta}{2}+\frac{\tilde{u}-\beta}{2}\sqrt{\frac{3}{5}}\right)}.$$

where $\tilde{u} = u_0+u_1+u_2+u_3$

This formulation allows us to suggest the following four step algorithm.

Algorithm 2(d): For a given u_0, approximate solution u_{n+1} is computed by the following iterative scheme:

$$y_n = u_n - \frac{f(u_n)}{f'(u_n)}, \tag{2. 30}$$

$$z_n = y_n - \frac{18f(y_n)}{5f' \quad \frac{y_n+u_n}{2} - \frac{y_n-u_n}{2}\sqrt{\frac{3}{5}} \quad + 8f' \quad \frac{y_n+u_n}{2} \quad + 5f' \quad \frac{y_n+u_n}{2} + \frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}}, \tag{2. 31}$$

$$w_n = z_n - \frac{18f(z_n)}{5f' \quad \frac{z_n+u_n}{2} - \frac{z_n-u_n}{2}\sqrt{\frac{3}{5}} \quad + 8f' \quad \frac{z_n+u_n}{2} \quad + 5f' \quad \frac{z_n+u_n}{2} + \frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}}, \tag{2. 32}$$

$$u_{n+1} = w_n - \frac{18f(w_n)}{5f' \quad \frac{w_n+u_n}{2} - \frac{w_n-u_n}{2}\sqrt{\frac{3}{5}} \quad + 8f' \quad \frac{w_n+u_n}{2} \quad + 5f' \quad \frac{w_n+u_n}{2} + \frac{w_n-u_n}{2}\sqrt{\frac{3}{5}}}, \qquad \mathrm{n} = 0, 1, 2, \ldots . \tag{2. 33}$$

3. CONVERGENCE ANALYSIS OF PROPOSED ALGORITHMS

In this section, we determine the convergence order of proposed Algorithm 2(b), Algorithm 2(c) and Algorithm 2(d).

Theorem 3.1. *For an open interval $I \subset R$. Let $f : I \to R$ and $\alpha \in I$ root of f(u)=0. If f is differentiable and u_0 is sufficiently close to α then two step method defined by Algorithm 2(b) has third order of convergence and satisfies the error equation:*

$$e_{n+1} = c_2^2 e_n^3 - \quad 3c_2^3 - 3c_2c_3 \quad e_n^4 + O(e_n^5).$$

Proof. Let α be simple zero of the function f then expanding $f(u_n)$ and $f'(u_n)$ about α

$$f(u_n) = f'(\alpha)\{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6\}, \tag{3. 34}$$

$$f'\big(u_n\big) = f'(\alpha)\{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6\}. \tag{3. 35}$$

where $c_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)}$ and $e_n = u_n - \alpha \quad \mathrm{k} = 2, 3, \ldots$

By using (3. 34), (3. 35) into (2. 24), we have

$$y_n = \alpha + c_2e_n^2 + \big(2c_3 - 2c_2^2\big)e_n^3 - \big(7c_2c_3 - 3c_4 - 4c_2^3\big)e_n^4 + \big(4c_5 - 10c_2c_4 - 6c_3^2 + 20c_3c_2^2 + 8c_2^4\big)e_n^5 + O(e_n^6), \tag{3. 36}$$

$$f(y_n) = f'(\alpha)\{c_2e_n^2 + \big(2c_3-2c_2^2\big)e_n^3 - \big(7c_2c_3 - 3c_4 - 4c_2^3\big)e_n^4 + \big(24c_3c_2^2 - 12c_2^4 - 6c_3^2 - 10c_2c_4 + 4c_5\}e_n^5 + O(e_n^6). \tag{3. 37}$$

Expanding $f'\Big(\frac{u_n+y_n}{2}\Big)$ $f'\big(\frac{y_n+u_n}{2} - \frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\big)$, $f'\big(\frac{y_n+u_n}{2} + \frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\big)$ by Taylor's series about α

$$f'\big(\frac{u_n+y_n}{2}\big) = f'(\alpha)\{1 + c_2e_n + \Big(c_2^2 + \frac{3c_3}{4}\Big) + \Big(\frac{7c_2c_3}{2} - 2c_2^3 + \frac{1}{2}c_4\Big)e_n^3 + \Big(4c_2^4 + \frac{9c_2c_4}{2} - \frac{37c_3c_2^2}{4} + \frac{15c_3^2c_5}{16}\Big)e_n^4\} + O(e_n^5), \tag{3. 38}$$

$$8f'\big(\frac{u_n+y_n}{2}\big) = f'(\alpha)\big\{8+8c_2e_n+\big(8c_2^2+6c_3\big)e_n^2+\big(28c_2c_3-16c_2^3+4c_4\big)e_n^3 + \big(\frac{5}{2}c_5+36c_2c_4-74c_3c_2^2+32c_2^4+24c_3^2\big)e_n^4+O(e_n^5), \qquad (3.\ 39)$$

$$f'\Big(\frac{y_n+u_n}{2}-\frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\Big) = f'(\alpha)\Big\{1+\big(5c_2+\frac{1}{5}c_2\sqrt{15}\big)e_n+\Big(c_2^2-\frac{1}{5}\sqrt{15}c_2^2+\frac{6}{5}c_3 + \frac{3}{10}\sqrt{15}c_3\Big)e_n^2+\big(\frac{13}{5}c_2c_3-2c_2^3+\frac{2}{5}\sqrt{15}c_2^3+\frac{7}{5}c_4 - \frac{2}{5}\sqrt{15}c_2c_3+\frac{9}{25}\sqrt{15}c_4\big)e_n^3+\Big(\frac{6}{5}c_3^2+\frac{11}{10}\sqrt{15}c_3c_2^2 - \frac{12}{25}\sqrt{15}c_2c_4+\frac{18}{5}c_2c_4+\frac{4}{5}\sqrt{15}c_2^4-7c_3c_2^2+4c_4 + \frac{31}{20}c_5+\frac{2}{5}\sqrt{15}c_5\Big)e_n^4+O(e_n^5)\Big\}, \qquad (3.\ 40)$$

$$5f'\Big(\frac{y_n+u_n}{2}-\frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\Big) = f'(\alpha)\Big\{5+\big(5c_2+c_2\sqrt{15}\big)e_n+\big(5c_2^2-\sqrt{15}c_2^2+6c_3 + \frac{3}{2}\sqrt{15}c_3\big)e_n^2+\big(13c_2c_3-10c_2^3+2\sqrt{15}c_2^3+7c_4 - 2\sqrt{15}c_2c_3+\frac{9}{5}\sqrt{15}c_4\big)e_n^3+\big(6c_3^2+\frac{11}{2}\sqrt{15}c_3c_2^2 + 18c_2c_4-\frac{12}{5}\sqrt{15}c_2c_4-4\sqrt{15}c_2^4-35c_3c_2^2+20c_4 + \frac{31}{4}c_5+2\sqrt{15}c_5\big)e_n^4+O(e_n^5)\Big\}, \qquad (3.\ 41)$$

$$5f'\Big(\frac{y_n+u_n}{2}+\frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\Big) = f'(\alpha)\Big\{5+\big(5c_2-c_2\sqrt{15}\big)e_n+\big(5c_2^2+\sqrt{15}c_2^2+6c_3 - \frac{3}{2}\sqrt{15}c_3\big)e_n^2+\big(13c_2c_3-10c_2^3-2\sqrt{15}c_2^3-\frac{9}{5}\sqrt{15}c_4 + 7c_4+2\sqrt{15}c_2c_3\big)e_n^3+\big(6c_3^2-\frac{11}{2}\sqrt{15}c_3c_2^2+18c_2c_4 + \frac{12}{5}\sqrt{15}c_2c_4+4\sqrt{15}c_2^4-35c_3c_2^2+20c_2^4-2\sqrt{15}c_5 + \frac{31}{4}c_5\big)e_n^4+O(e_n^5)\Big\}. \qquad (3.\ 42)$$

From (3. 37), (3. 39), (3. 41), (3. 42), we get

$$\frac{f(y_n)}{5f'\big(\frac{y_n+u_n}{2}-\frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\ \big)+8f'\big(\frac{y_n+v_n}{2}\big)+f'\big(\frac{y_n+u_n}{2}-\frac{y_n-u_n}{2}\sqrt{\frac{3}{5}}\big)} = c_2e_n^2+\big(2c_3-3c_2^2\big)e_n^3+O(e_n^4). \qquad (3.\ 43)$$

Substituting values from (3. 36), (3. 43) into (2. 25) and simplifying we get the error term of the Algorithm 2(b)

$$u_{n+1} = \alpha + c_2^2 e_n^3 - \big(3c_2^3 - 3c_2c_3\big)e_n^4 + O(e_n^5), \tag{3. 44}$$

$$e_{n+1} = c_2^2 e_n^3 - \big(3c_2^3 - 3c_2c_3\big)e_n^4 + O(e_n^5). \tag{3. 45}$$

□

Theorem 3.2. *For an open interval $I \subset R$.Let $f : I \to R$ and $\alpha \in I$ be root of $f(u) = 0$. If f is differentiable and u_0 is sufficiently close to α then three step method defined by Algorithm 2(c) has fourth order of convergence and it satisfies the error equation:*

$$e_{n+1} = c_2^3 e_n^4 + (4c_3c_2^2 - 4c_2^4)e_n^5 + O(e_n^6). \tag{3. 46}$$

Proof. Considering again the equation (3. 44), we have

$$z_n = \alpha + c_2^2 e_n^3 + \big(-3c_2^3 + 3c_2c_3\big)e_n^4 + \big(-12c_3c_2^2 + 6c_2^4 + 4c_2c_4 + 2c_3^2\big)e_n^5 + O(e_n^6), \tag{3. 47}$$

$$f\big(z_n\big) = f'(\alpha)\Big\{c_2^2 e_n^3 + \big(3c_2c_3 - 3c_2^3\big)e_n^4 + \big(-12c_3c_2^2 + 2c_3^2 + 6c_2^4 + 4c_2c_4\big)e_n^5 + O(e_n^6)\Big\}. \tag{3. 48}$$

Expanding $f'\big(\frac{u_n+z_n}{2}\big)$, $f'\big(\frac{z_n+u_n}{2} - \frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\big)$, $f'\big(\frac{z_n+u_n}{2} + \frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\big)$ by Taylor series about α

$$\begin{aligned} 8f'\big(\frac{u_n+z_n}{2}\big) =& f'(\alpha)\Big\{8 + 8c_2e_n + 6c_3e_n^2 + \big(8c_2^3 + 4c_4\big)e_n^3 + \big(-24c_2^4 + 36c_3c_2^2 \\ &+ \frac{5}{2}c_5\big)e_n^4 - \big(132c_2c_2^3 - 52c_2c_3^2 - \frac{3}{2}c_6 - 44c_4c_2^2 - 48c_2^5\big)e_n^5 + O(e_n^5)\Big\}, \end{aligned} \tag{3. 49}$$

$$\begin{aligned} 5f'\Big(\frac{z_n+u_n}{2} - \frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\Big) =& f'(\alpha)\Big\{5 + \big(5c_2 + c_2\sqrt{15}\big)e_n + \big(6c_3 + \frac{3}{2}\sqrt{15}c_3\big)e_n^2 \\ &+ \big(5c_2^3 - \sqrt{15}c_2^3 + 7c_4 + \frac{9}{5}\sqrt{15}c_4\big)e_n^3 - \big(15c_2^4 - 18c_3c_2^2 \\ &- 3\sqrt{15}c_2^4 + 3\sqrt{15}c_3c_2^2 - \frac{31}{4}c_5 - 2\sqrt{15}c_5\big)e_n^4 + O(e_n^5)\Big\}, \end{aligned} \tag{3. 50}$$

$$\begin{aligned} 5f'\Big(\frac{z_n+u_n}{2} + \frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\Big) =& f'(\alpha)\Big\{5 + \big(5c_2 - c_2\sqrt{15}\big)e_n + \big(6c_3 - \frac{3}{2}\sqrt{15}c_3\big)e_n^2 \\ &+ \big(5c_2^3 + \sqrt{15}c_2^3 + 7c_4 - \frac{9}{5}\sqrt{15}c_4\big)e_n^3 - \big(15c_2^4 - 18c_3c_2^2 \\ &+ 3\sqrt{15}c_2^4 - 3\sqrt{15}c_3c_2^2 - \frac{31}{4}c_5 + 2\sqrt{15}c_5\big)e_n^4 + O(e_n^5)\Big\}. \end{aligned} \tag{3. 51}$$

From (3. 48), (3. 49) (3. 50), (3. 51) , we have

$$\frac{f(z_n)}{5f'\big(\frac{z_n+u_n}{2}-\frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\big)+8f'\big(\frac{u_n+y_n}{2}\big)+5f'\big(\frac{z_n+u_n}{2}+\frac{z_n-u_n}{2}\sqrt{\frac{3}{5}}\big)} = c_2^2e_n^3-\big(4c_2^3-3c_2c_3\big)e_n^4-\big(16c_3c_2^2-10c_2^4-4c_2c_4-2c_3^2\big)e_n^5+O(e_n^6). \quad (3.\ 52)$$

Substituting values from (3. 47), (3. 52) into (2. 29) and simplifying we get the error term of the Algorithm 2(c)

$$u_{n+1}=\alpha+c_2^3e_n^4+\big(4c_3c_2^2-4c_2^4\big)e_n^5+O(e_n^6), \quad (3.\ 53)$$

$$e_{n+1}=c_2^3e_n^4+\big(4c_3c_2^2-4c_2^4\big)e_n^5+O(e_n^6). \quad (3.\ 54)$$

□

Theorem 3.3. *For an open interval $I\subset R$. Let $f: I\to R$ and $\alpha\in I$ be root of $f(u)=0$. If f is differentiable and u_0 is sufficiently close to α then four step method defined by Algorithm 2(d) has fifth order of convergence and it satisfies the error equation:*

$$e_{n+1}=c_2^4e_n^5+\big(5c_3c_2^3-5c_2^5\big)e_n^6+O(e_n^7). \quad (3.\ 55)$$

Proof. Consider the equation (3. 53)

$$w_n=\alpha+c_2^3e_n^4+\big(4c_3c_2^2-4c_2^4\big)e_n^5+\big(5c_2^2c_4+5c_2c_3^2-20c_3c_2^2+10c_2^5\big)e_n^6+O(e_n^7). \quad (3.\ 56)$$

Expanding $f(w_n)$ by Taylor's series about α

$$f(w_n)=f'(\alpha)\{c_2^3e_n^4+\big(4c_3c_2^2-4c_2^4\big)e_n^5+\big(5c_4c_2^2+5c_2c_3^2-20c_3c_2^3+10c_2^5\big)e_n^6+O(e_n^7). \quad (3.\ 57)$$

Expanding $f'\Big(\frac{w_n+u_n}{2}\Big), f'\Big(\frac{w_n+u_n}{2}-\frac{w_n-u_n}{2}\sqrt{\frac{3}{5}}\Big), f'\Big(\frac{w_n+u_n}{2}+\frac{w_n-u_n}{2}\sqrt{\frac{3}{5}}\Big)$ in terms of Taylor's series about α

$$8f'\big(\frac{w_n+u_n}{2}\big)=f'(\alpha)\{8+8c_2e_n+6c_3e_n^2+4c_4e_n^3+\big(8c_2^4+\frac{5}{2}c_5\big)e_n^4+\big(\frac{3}{2}c_6+44c_3c_2^3-32c_2^5\big)e_n^5+O(e_n^6), \quad (3.\ 58)$$

$$\begin{aligned}5f'\big(\frac{w_n+u_n}{2}-\frac{w_n-u_n}{2}\sqrt{\frac{3}{5}}\big)=&f'(\alpha)\{5+\big(5c_2+c_2\sqrt{15}\big)e_n+\big(6c_3+\frac{3}{2}\sqrt{15}c_3\big)e_n^2\\&+\big(7c_4+\frac{9}{5}\sqrt{15}c_4\big)e_n^3+\big(2\sqrt{15}c_5-\sqrt{15}c_2^4+5c_2^4\\&+\frac{31}{4}c_5\big)e_n^4+\big(\frac{33}{4}c_6-4\sqrt{15}c_3c_2^3+4\sqrt{15}c_2^5+23c_3c_2^3\\&+\frac{213}{100}\sqrt{15}c_6-20c_2^5\big)e_n^5+O(e_n^6),\end{aligned} \quad (3.\ 59)$$

$$5f'\Big(\frac{w_n+u_n}{2}+\frac{w_n-u_n}{2}\sqrt{\frac{3}{5}}\Big)=f'(\alpha)\Big\{5+\big(5c_2-c_2\sqrt{15}\big)e_n+\big(6c_3-\frac{3}{2}\sqrt{15}c_3\big)e_n^2 + \big(7c_4-\frac{9}{5}\sqrt{15}c_4\big)e_n^3+\big(-2\sqrt{15}c_5+\sqrt{15}c_2^4+5c_2^4 + \frac{31}{4}c_5\big)e_n^4+\big(-4\sqrt{15}c_3c_2^3-4\sqrt{15}c_2^5+23c_3c_2^3 - 20c_2^5+\frac{33}{4}c_6-\frac{213}{100}\sqrt{15}c_6\big)e_n^5+O(e_n^6). \quad (3.\ 60)$$

From (3. 57), (3. 58), (3. 59), (3. 60) we have

$$\frac{f(w_n)}{5f'\big(\frac{w_n+u_n}{2}-\frac{w_n-u_n}{2}\sqrt{\frac{3}{5}}\big)+8f'\big(\frac{w_n+u_n}{2}\big)+5f'\big(\frac{w_n+u_n}{2}+\frac{w_n-u_n}{2}\sqrt{\frac{3}{5}}\big)} = c_2^3e_{n+}^4\big(4c_3c_2^2-5c_2^4\big)e_n^5+O(e_n^6). \quad (3.\ 61)$$

Substituting values from (3. 56), (3. 61) into (2. 33) and simplifying we get the error term of the Algorithm 2(d)

$$u_{n+1}=\alpha+c_2^4e_n^5+\big(5c_3c_2^3-5c_2^5\big)e_n^6+O(e_n^7), \quad (3.\ 62)$$

$$e_{n+1}=c_2^4e_n^5+\big(5c_3c_2^3-5c_2^5\big)e_n^6+O(e_n^7). \quad (3.\ 63)$$

□

The error equations (3. 45), (3. 54), and (3. 63) show that the Algorithm 2(b), Algorithm 2(c) and Algorithm 2(d) are third fourth $\&$ fifth order convergent.
This completes the proof.

4. NUMERICAL EXAMPLES AND APPLICATIONS

In this section, we consider some nonlinear models related to Mathematics and Engineering which include continuous stirred tank reactor problem, van der Waal's equation, renowned population growth model, and nonlinear model formed due to motion of a particle on an inclined plane. We also include some of the nonlinear equations used by Chun [7] to elaborate the efficacy and effectiveness of the proposed algorithms. We obtain estimated simple root rather than the exact depending on the exactness $\varepsilon = 10^{-100}$ of the computer. For computational work, we implement codes in Maple software and Matlab for graphical analysis, and the following stopping criterion is taken into account for entire computations:

$$(i).|u_{n+1}-u_n|<\varepsilon \qquad and \qquad (ii).|f(u_{n+1}|<\varepsilon. \quad (4.\ 64)$$

We compare the Newton-Raphson method (NM), Chun method (CM) [7], Weerakoon and Fernando method (WF) [34], Noor et al. methods [26] $\{$Algorithm 2.5 (NR1), Algorithm 2.8 (NR2), Algorithm 2.10 (NR3)$\}$ with the Algorithm 2(b) (AG1), Algorithm 2(c) (AG2), Algorithm 2(d)(AG3). As for the convergence criteria, it was required that the distance of two consecutive approximations (δ) for the zero was less than 10^{-100}. We display the number of iterations (IT), the approximate zero u_n and the functional value $f(u_n)$ in Tables

(1– 5). The computational order of convergence (COC) (see [8]) is computed to check the behaviour of the proposed methods for presented examples and given by:

$$\text{COC} \approx \frac{ln|\ u_{n+1} - u_n\ |/|\ u_n - u_{n-1}\ |}{ln|\ u_n - u_{n-1}\ |/|\ u_{n-1} - u_{n-2}\ |}. \tag{4. 65}$$

Example 4.1 (see [5](Continuous stirred tank reactor (CSTR))**.** *Here, we consider the nonlinear equation stirred tank reactor i.e.,*

$$u^4 + 11.50u^3 + 47.49u^2 + 83.06325u + 51.23266875 = 0. \tag{4. 66}$$

We use $x_o = -1.40$ as an initial guess for the computer programs in this example. The comparison of numerical results for this example is given in Table 1. It is clear from the numerical results obtained from Table 1 that the effectiveness and presentation of the new methods are much better than those of the other similar order methods. Fig 1 also shows the efficacy of proposed schemes by making the comparison of the iterative methods concerning to different methods in terms of the number of iterations.

Table 1: Numerical results for Equation 4. 66 .

Methods	IT	u_n	$f(u_n)$	$\delta = \|x_n - x_{n-1}\|$	COC
NM	6	1.4500000000000000	3.258864e-69	1.798055e-35	2.00000
WF	4	1.4500000000000000	4.450403e-85	2.779472e-29	2.99999
NR1	4	1.4500000000000000	1.034328e-76	1.424909e-26	2.99973
NR2	4	1.4500000000000000	1.545514e-87	4.421829e-30	2.99992
NR3	3	1.4500000000000000	8.507087e-58	1.609392e-15	3.93555
CM	3	1.4500000000000000	3.887894e-56	3.957471e-15	3.92523
AG1	3	1.4500000000000000	2.513368e-29	1.120288e-10	2.99999
AG2	3	1.4500000000000000	3.280405e-69	3.189433e-18	3.96872
AG3	2	1.4500000000000000	3.605181e-27	2.298321e-06	4.97036

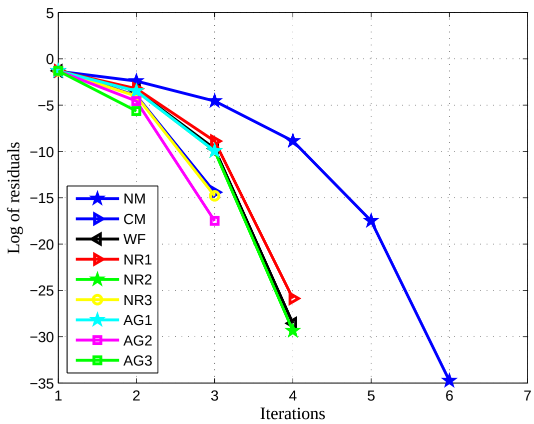

Figure 1: Log of residuals for Equation 4. 66 .

Example 4.2 (Van der Waal's equation see [33])**.** *We consider the van der Waal's qquation representing the real and ideal behaviour of gass is prescribed as:*

$$\left(\mathcal{Q}+\frac{cp^2}{V^2}\right)(\mathcal{V}-pd)=p\mathcal{S}\mathcal{T}, \tag{4. 67}$$

equation.(4. 67) can be transformed to the following nonlinear form:

$$\mathcal{Q}\mathcal{V}^3-(pd\mathcal{Q}+p\mathcal{S}\mathcal{T})\mathcal{V}^2+cp^2\mathcal{V}-cp^3d, \tag{4. 68}$$

after approperiatly choosing the needed parametres and unknown constants we can find the following nonlinear function:

$$0.986u^3-5.181u^2+9.067u-5.289=0. \tag{4. 69}$$

Where the variable $\tilde{u}$ shows the volume of the gas. We take $\tilde{u}_0=3.10$ as an initial guess for computational evaluations. The mathematical computations for Eq.(4. 69) are calculated in Table 2. Fig 2 shows the comparison of the iterative methods with respect to the number of iterations.

Table 2: Numerical results for Equation 4. 69 .

Methods	IT	u_n	$f(u_n)$	$\delta=\lvert x_n-x_{n-1}\rvert$	COC
NM	12	1.9298462428478622	3.825655e-94	2.693071e-47	2.00000
WF	8	1.9298462428478622	1.000000e-126	5.142285e-43	3.00000
NR1	8	1.9298462428478622	5.845408e-80	2.086153e-27	2.99975
NR2	8	1.9298462428478622	0.000000e+00	3.271773e-50	3.00000
NR3	7	1.9298462428478622	1.000000e-126	8.979952e-39	3.99905
CM	7	1.9298462428478622	0.000000e+00	8.145024e-35	3.99798
AG1	7	1.9298462428478622	2.827856e-51	9.577165e-18	2.99612

AG2	6	1.9298462428478622	2.541997e-93	3.372664e-24	3.99999
AG3	5	1.9298462428478622	1.807906e-71	2.726914e-15	4.96276

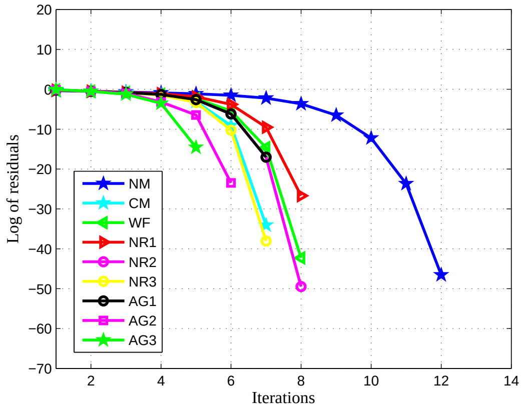

Figure 2: Log of residuals for Equation 4. 69 .

Table 2 and Figure 2 illustrate that the new methods, $AG1$, $AG2$, $AG3$ require less number of iterations when compared with NM, CM, WF, $NR1$, $NR2$, $NR3$ to meet the stopping criterium (4. 64). It is clear from the results of this example that the proposed methods converge more rapidly to the solution as compared to the already existed methods.

Example 4.3 (see [6](Population growth model)). *Consider the nonlinear equation that appears in mathematical modeling of the growth of population over short periods of time:*

$$1564,000 = 1000,000e^{\lambda} + 435000\Big(\frac{e^{\lambda} - 1}{\lambda}\Big) - 1564000, \tag{4. 70}$$

where λ denotes the constant birth rate of population whose value needs to determined. For computational work, we take $x_o = 1.5$ as an initial estimate. The numerical results for this problem are given in Table 3. Fig 3 show the comparison of the iterative methods with respect to number of iterations.

Table 3: Numerical results for Equation 4. 70 .

Methods	IT	u_n	$f(u_n)$	$\delta = \lvert x_n - x_{n-1} \rvert$	COC
NM	7	0.1009979296857498	7.697779e-31	1.104193e-18	2.00000
WF	5	0.1009979296857498	3.049742e-54	1.968666e-20	2.99999
CM	5	0.1009979296857498	3.000000e-121	7.646560e-45	3.99979
NR1	5	0.1009979296857498	5.620211e-45	2.113407e-17	2.99775
NR2	4	0.1009979296857498	1.334042e-63	1.648664e-23	2.99979
NR3	4	0.1009979296857498	3.875494e-41	2.882365e-12	3.92829
AG1	4	0.1009979296857498	2.200930e-17	4.197052e-08	2.97866
AG2	4	0.1009979296857498	3.380718e-67	1.245760e-18	3.98809
AG3	3	0.1009979296857498	7.972330e-30	1.644990e-07	4.96899

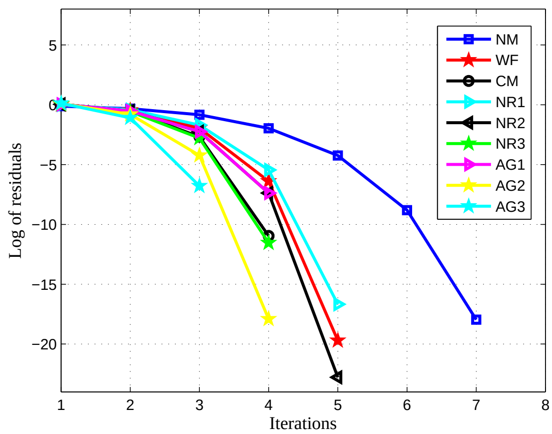

Figure 3: Log of residuals for Equation 4. 70 .

The second column of Table 3 demonstrates that new methods require less number of iteration to reach the stopping criteria (4. 64) and the same for some cases when compared with different methods. It is clear from Table 3 and Fig 3 the approximate solutions obtained by the newly proposed methods can be compared with the well-known existing methods and performance is much better than the other methods. Computational order of convergence is also exhibited in the Table 3 which also verifies the convergence order of the methods.

Example 4.4 (see [26](Motion of particle on an Inclined plane)).

We solve the nonlinear model formed due to the motion of a particle on an inclined plane whose angle of inclination, θ changes at a constant rate $\frac{d(\theta)}{dt} = w < 0$:

$$x(t) = -\frac{g}{2w^2}\Big(\frac{e^{wt} - e^{-wt}}{2} - sinwt\Big). \qquad (4.\ 71)$$

We take $x_o = -0.36$ as an initial guess for computation evaluation. The comparison of numerical results for this problem are given in Table 4. Figure 4 represent the fall of residuals for this example.

Table 4: Numerical results for Equation 4. 71 .

Methods	IT	x_n	$f(u_n)$	$\delta = \|x_n - x_{n-1}\|$	COC
NM	6	0.3170617745729571	1.402507e-44	2.575729e-22	1.99999
WF	4	0.3170617745729571	7.807920e-54	1.732945e-18	2.99671
NR1	4	0.3170617745729571	1.360230e-46	3.708416e-16	2.99170
NR2	4	0.3170617745729571	1.292575e-55	4.593526e-19	2.99721
NR3	3	0.3170617745729571	4.197484e-35	1.056817e-09	3.71019
CM	3	0.3170617745729571	7.409941e-34	2.048702e-09	3.99675
AG1	4	0.3170617745729571	1.292567e-55	4.593517e-19	2.99720
AG2	3	0.3170617745729571	1.445782e-44	6.438628e-12	3.99957
AG3	3	0.3170617745729571	3.562374e-84	9.233781e-18	4.99999

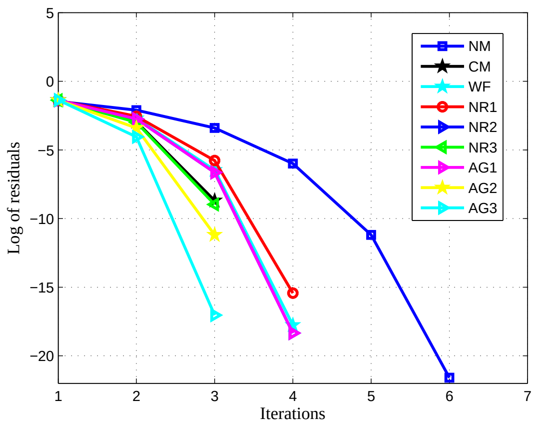

Figure 4: Log of residuals for Equation 4. 71 .

It is clear from computational results for the effectiveness and presentation of the new methods are much better and needs less number of iterations when compared with NM, NR1, NR2, NR3, CM, WF or same in cases (CM, NR3). In Table 4, and Fig 4 show the compatibility of numerical results for new iterative methods with the methods theoratical analysis.

Example 4.5 (Transcendental and Algebraic problems.)**.**

To numerically analyze the suggested algorithms, we consider the following transcendental and algebraic equations used by Chun [7]:

$$\begin{aligned} f_1(u) =& (u-1)^3 - 1, & u_o = 3.5,\\ f_2(u) =& cos(u) - u, & u_o = 1.7,\\ f_3(u) =& e^{u^2+7u-30} - 1, & u_o = 3.5,\\ f_4(u) =& sin^2(u) - u^2 + 1, & u_o = -1,\\ f_5(u) =& u^2 - e^u - 3u + 2, & u_o = 2. \end{aligned}$$

In Table 5, we display the numerical results for examples $f_1(u)$, $f_2(u)$, $f_3(u)$, $f_4(u)$, $f_5(u)$ to validate the theoretical results.

Table 5: Numerical comparison between different algorithms for test problems $f_1(s)$–$f_5(s)$.

Methods	IT	u_n	$f(u_n)$	$\delta = \lvert u_n - u_{n-1}\rvert$	COC
$f_1(u) = (u-1)^3 - 1, \quad u_o = 3.5$					
NM	8	2.0000000000000000	2.056946e-42	8.280390e-22	2.00000
CM	6	2.0000000000000000	0.000000e+00	2.827696e-94	4.00000
WF	5	2.0000000000000000	9.839450e-37	6.550899 e-13	2.98641
NR1	7	2.0000000000000000	0.000000e+00	1.240603e-83	2.99999
NR2	6	2.0000000000000000	4.040313e-120	1.104328e-40	3.00000
NR3	5	2.0000000000000000	0.000000e+00	5.864694e-26	3.99224
AG1	5	2.0000000000000000	3.312983 e-40	4.797695 e-14	2.99061
AG2	4	2.0000000000000000	2.932850e-42	3.144431e-11	3.86663
AG3	3	2.0000000000000000	1.015348e-20	8.052339e-05	4.86465
$f_2(u) = cos(u) - u, \quad u_0$= 1.7,					
NM	7	0.7390851332151608	0.000000e+00	3.254881e-65	2.00000
WF	5	0.7390851332151606	0.000000e+00	1.696999e-65	3.00000
NR1	5	0.7390851332151608	0.000000e+00	5.225607e-73	3.00000
NR2	5	0.7390851332151606	0.000000e+00	6.108403e-80	3.00000
NR3	4	0.7390851332151606	0.000000e+00	2.241483e-55	3.99987
CM	4	0.7390851332151606	0.000000e+00	1.867468e-53	3.99983
AG1	4	0.7390851332151606	9.975704e-80	1.069276e-26	2.99942
AG2	4	0.7390851332151620	0.000000e+00	4.125693e-62	3.99995
AG3	2	0.7390851332151620	7.930209e-24	7.243271e-05	4.88893
$f_3(u) = e^{u^2+7u-30} - 1, \quad u_0$= 3.5					
NM	13	3.0000000000000000	1.516930e-47	4.212111e-25	2.00000
CM	9	3.0000000000000000	0.000000e+00	2.884803e-88	4.00000
WF	9	3.0000000000000000	5.066487e-71	4.069243e-25	2.99979

NR1	10	3.0000000000000000	0.000000e+00	3.108113e-46	3.00000
NR2	8	3.0000000000000000	7.833171e-38	5.183816e-14	2.99434
NR3	9	3.0000000000000002	0000000e+00	2.495598e-99	3.99999
AG1	8	3.0000000000000510	7.473796e-38	5.103296e-14	2.99999
AG2	7	3.0000000000000000	7.004102e-93	1.173100e-24	3.99523
AG3	6	3.0000000000000000	1.159203e-87	5.440327e-19	4.95234
$f_4(u) = sin^2(u) - u^2 + 1$ u_0= -1					
NM	7	1.4044916482153412	1.044527e-50	7.327881e-26	2.00000
CM	6	1.4044916482153412	2.100000e-127	7.126555e-68	3.99999
WF	5	1.4044916482153412	8.904296e-89	3.791998e-30	3.00010
NR1	16	1.4044916482153412	1.095509e-20	1.531728e-07	2.97384
NR2	5	1.4044916482153412	5.953595e-94	7.309921e-32	3.00005
NR3	5	1.4044916482153412	1.000000e-127	6.860606e-97	3.99935
AG1	4	1.4044916482153412	1.773310e-31	4.881813e-11	3.03474
AG2	3	1.4044916482153412	7.210014e-25	8.814637e-07	4.33293
AG3	3	1.4044916482153412	3.720234e-48	3.310354e-10	4.99921
$f_5(u) = u^2 - e^u - 3u + 2,$ u_0= 2					
NM	7	0.2575302854398608	2.117415e-111	7.743422e-56	2.00000
WF	5	0.2575302854398608	5.935509e-103	1.630507e-11	3.00999
NR1	4	0.2575302854398608	1.218871e-41	5.694586e-14	2.91743
NR2	4	0.2575302854398608	3.376128e-53	1.007610e-17	2.97840
CM	4	0.2575302854398608	1.236660e-114	9.463123e-29	3.97784
NR3	3	0.2575302854398608	9.720474e-29	2.979348e-07	4.65461
AG1	3	0.2575302854398608	3.732105e-17	1.041853e-05	3.48762
AG2	3	0.2575302854398607	3.162123e-45	3.182065e-11	4.14451
AG3	2	0.2575302854398607	2.735843e-18	1.569721e-03	4.94443

The second column in Table 5 represent the number of iterations for different nonlinear functions along with initial guess u_0. A comparative representation of the number of iterations is presented, needed for different methods with our developed methods using the stopping criteria (4. 64) $\&$ accuracy $\varepsilon = 10^{-100}$. It is clear from Table 5 that settling the same convergence criteria for all the methods, the number of iterations required for the new methods remains less or equal in some cases than the number of iterations needed by the other methods of the same order. The computational order of convergence is computed to validate theoratical results and it varies for all the methods.

5. Conclusion

In this manuscript, we have presented the new third, fourth-order, and fifth-order convergent iterative schemes by using the decomposition technique and quadrature formula. The efficiency and performance of these schemes are checked with the support of some numerical examples. A comparative analysis of these newly established schemes with some known schemes existing in the literature is also presented to validate the theoretical results. The computational numerical results and graphical analysis show the supermacy of the proposed schemes over the other iterative schemes; see Tables 1–5 and Figures 1–4.

CONFLICT OF INTEREST

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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