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Generalized Functional Equations of Polylogarithmic Groups

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Abstract. In this article, the functional equations of classical polylogarithmic groups for higher weights have been introduced and their relations with configuration of complexes have been defined via morphisms. Furthermore, these functional equations and homomorphisms have been generalized and their proofs have also been given. Through the use of these functional equations, the bi-complex form of the associated diagrams has been shown.

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1. Introduction

Classical logarithmic functions have been studied for many years and have specially been used in quantum electrodynamics, quantum statistics, series representations and many other similar physical phenomena. The dilogarithmic form of such functions was defined by Leibniz whereas the key element of Goncharov's work was proposing a triple-ratio for tri-logarithmic groups [4], then generalizing these groups as $\mathcal{B}_n(F)$ by framing them as Goncharov's complex. Further, remarkable contribution was to derive the functional equations for weight 2 and 3 for bilogarithmic and trilogarithmic groups $\mathcal{B}_2(F)$ and $\mathcal{B}_3(F)$ [2–4]. Moreover, it was proved that the associative chain complexes are commutative (see [2]).
In the work of [14], the functional equation of weight 4 for the group $\mathcal{B}_4(F)$ is introduced, those were 9 terms relations. Khalid et al. [9, 10] introduced some extension and generalization in the geometry of Grassmannian configuration and variant of Cathelineau chain

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complexes. Khalid et al. [8, 11, 12] also extended the geometry of Goncharov classical polylogarithmic group and configured the chain complex.
In this work, the generalization of these functional equations have been proven for weight 5,6 upto weight-n. Also, it is shown that the resultant diagrams are bi-complex. In short, this work is generalization and the extension of Goncharov.

2. Basic Concepts and Background

2.1. **Generalized Grassmannian Configuration Chain Complex.** Let consider the following generalized Grassmannian configuration chain complex (see [5–7, 13])

$$\begin{tikzcd} G_{n+5}(n+2) \arrow[r, "d"] \arrow[d, "p"] & G_{n+4}(n+2) \arrow[r, "d"] \arrow[d, "p"] & G_{n+3}(n+2) \arrow[d, "p"] \\ G_{n+4}(n+1) \arrow[r, "d"] \arrow[d, "p"] & G_{n+3}(n+1) \arrow[r, "d"] \arrow[d, "p"] & G_{n+2}(n+1) \arrow[d, "p"] \\ G_{n+3}(n) \arrow[r, "d"] & G_{n+2}(n) \arrow[r, "d"] & G_{n+1}(n) \end{tikzcd} \tag{A}$$

where the group $G_{n+1}(n)$ be a free abelian group. It is generated by the configuration of $(n+1)$-vectors in n dimensional vector space. Following are two type of differential morphisms.

$$d : (\kappa_0, \ldots, \kappa_n) \mapsto \sum_{i=0}^{n} (-1)^i (\kappa_0, ..., \hat{\kappa}_i, ..., \kappa_n) \tag{2.1}$$

and

$$p : (\kappa_0, \ldots, \kappa_n) \mapsto \sum_{i=0}^{n} (-1)^i (\kappa_i | \kappa_0, \ldots, \hat{\kappa}_i, \ldots, \kappa_n) \tag{2.2}$$

Lemma 2.2. *The generalized diagram* A *is bi-complex, see* [13].

Lemma 2.3. *The generalized diagrame* A *is commutative, see* [13].

2.4. **Polylogarithmic Groups of Weight 1.** Let consider a $Z-$ module denoted by $Z[\mathbf{P}^1_F]$. It is generated by symbol $[x] \in \mathbf{P}^1_F$ [14]. We will use F as a field and $F^{\bullet\bullet} = F - \{0, 1\}$ is a double punctured set. The polylogarithmic group for weight 1 is denoted by $\mathcal{B}(F)$ is a scissor congruence group [1] which is quotient group of $Z[F^{\bullet\bullet}]$ and its subgroup generated by the famous Abel's famous five term relation,

$$[u] - [v] + [\frac{v}{u}] - [\frac{1-v}{1-u}] + [\frac{1-v^{-1}}{1-u^{-1}}]$$

where $u \neq v$,$u, v \neq 0, 1$ (see [2])

2.4.1. *Bloch Groups, the Bloch-Suslin complex and Generalized Goncharov's Polylogarithmic Complexes.*
Weight 1. Consider the subgroup defined as $R_1(F) \subset Z[\mathbf{P}^1_F]$ generated by the relation $[uv] - [u] - [v]$ called the three term relation where $u, v \in F^\times$ such that $R_1(F) = \mathcal{B}_1(F)$. Defined the an isomorphism $\delta : \mathcal{B}_1(F) \to F^\times, [x] \to x$(see [2]). Also $\mathcal{B}_1(F) = F^\times$

Weight 2. Let the introduced subgroup $R_2(F) \subset Z[\mathbf{P}^1_F]$ (see [2]) generated by cross ratio of five terms be defined as

$$\sum_{i=0}^{4}(-1)^i[r[\kappa_0, ..., \hat{\kappa}_i, ..., \kappa_4] \tag{2.3}$$

where

$$r(\kappa_0, ..., \kappa_3) = \frac{\triangle(\kappa_0, \kappa_3)\triangle(\kappa_1, \kappa_2)}{\triangle(\kappa_0, \kappa_2)\triangle(\kappa_1, \kappa_3)}$$

By introducing a map $\delta_2 : Z[\mathbf{P}^1_F/\{0, 1, \infty\}] \to \wedge^2 F^\times$, introduced as $[u] \to (1 - u) \wedge u$, it is also shown that the composition $\delta_2(R_2(F)) = 0$ (see [2]). Now, introduced the group denoted by $\mathcal{B}_2(F)$. It is the quotient group of module and its subgroup $Z[\mathbf{P}^1_F/\{0, 1, \infty\}]/R_2(F)$. Let us introduce the Bloch-Suslin complex

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times \ .$$

The morphism δ is given as $\delta : [u]_2 \to (1 - u) \wedge u$
Weight 3. As formalized in [2]

$$r_3(\kappa_0, ..., \kappa_6) = Alt_6 \frac{\triangle(\kappa_0, \kappa_1, \kappa_3)\triangle(\kappa_1, \kappa_2, \kappa_4)\triangle(\kappa_2, \kappa_0, \kappa_5)}{\triangle(\kappa_0, \kappa_1, \kappa_4)\triangle(\kappa_1, \kappa_2, \kappa_5)\triangle(\kappa_2, \kappa_0, \kappa_3)},$$

it is also called the seven term relation. Let consider a subgroup $R_3(F) \in Z[\mathbf{P}^1_F]$ [2] is defined as

$$R_3(F) = \sum_{i=0}^{6}(-1)^i r_3(\kappa_0, ..., \hat{\kappa}_i, ..., \kappa_6) \tag{2.4}$$

which is a seven term relation of the triple ratio. Goncharov defined $\mathcal{B}_3(F)$ as quotient subgroup $Z[\mathbf{P}^1_F]/R_3(F)$, then the Goncharov's polylogarithmic chain complex for weight 3 is given by

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^\times \xrightarrow{\delta} \wedge^3 F^\times$$

Therefore the functional equations of $\mathcal{B}_3(F)$ are seven terms relation.
Weight 4. Suppose

$$r_4(\kappa_0, ..., \kappa_7) = Alt_8 \frac{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_4)\triangle(\kappa_1, \kappa_2, \kappa_3, \kappa_5)\triangle(\kappa_2, \kappa_3, \kappa_0, \kappa_6)\triangle(\kappa_3, \kappa_0, \kappa_1, \kappa_7)}{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_5)\triangle(\kappa_1, \kappa_2, \kappa_3, \kappa_6)\triangle(\kappa_2, \kappa_3, \kappa_0, \kappa_7)\triangle(\kappa_3, \kappa_0, \kappa_1, \kappa_4)}$$

and

$$R_4(F) = \sum_{i=0}^{8}(-1)^i r_4(\kappa_0, ..., \hat{\kappa}_i, ..., \kappa_8), \tag{2.5}$$

it is also known as a nine term relation. Now by defining the Bloch group for weight 4 denoted by $\mathcal{B}_4(F)$ which is quotient subgroup as $Z[\mathbf{P}^1_F]/R_4(F)$. Hence, functional equations of $\mathcal{B}_4(F)$ is a 9 term relation in which total numbers of terms are $9(7!) = 45360$

Weight 5. By introducing

$$r_5(\kappa_0, ..., \kappa_9) = Alt_{10} \frac{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_5)\triangle(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_6)\triangle(\kappa_2, \kappa_3, \kappa_4, \kappa_0, \kappa_7)}{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_6)\triangle(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_7)\triangle(\kappa_2, \kappa_3, \kappa_4, \kappa_0, \kappa_8)}. \frac{\triangle(\kappa_3, \kappa_4, \kappa_0, \kappa_1, \kappa_8)\triangle(\kappa_4, \kappa_0, \kappa_1, \kappa_2, \kappa_9)}{\triangle(\kappa_3, \kappa_4, \kappa_0, \kappa_1, \kappa_9)\triangle(\kappa_4, \kappa_0, \kappa_1, \kappa_2, \kappa_5)} \tag{2.6}$$

and also

$$R_5(F) = \sum_{i=0}^{10} (-1)^i r_5(\kappa_0, ..., \hat{\kappa}_i, ..., \kappa_{10}). \tag{2.7}$$

The above relation is an eleven term relation. So define $\mathcal{B}_5(F) = Z[\mathbf{P}_F^1]/R_5(F)$. Therefore, functional equations of $\mathcal{B}_5(F)$ is an eleven term relation in which total number of terms are $11(9!) = 3991680$

Weight 6. Similarly, if it has 12 points then

$$r_6 = Alt_{12} \frac{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_6)\triangle(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_7)\triangle(\kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_0, \kappa_8)}{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_7)\triangle(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_8)\triangle(\kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_0, \kappa_9)}. \frac{\triangle(\kappa_3, \kappa_4, \kappa_5, \kappa_0, \kappa_1, \kappa_9)\triangle(\kappa_4, \kappa_5, \kappa_0, \kappa_1, \kappa_2, \kappa_{10})\triangle(\kappa_5, \kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_{11})}{\triangle(\kappa_3, \kappa_4, \kappa_5, \kappa_0, \kappa_1, \kappa_{10})\triangle(\kappa_4, \kappa_5, \kappa_0, \kappa_1, \kappa_2, \kappa_{11})\triangle(\kappa_5, \kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_6)} \tag{2.8}$$

and also

$$R_6(F) = \sum_{i=0}^{13} (-1)^i r_6(\kappa_0, ..., \kappa_{12}) \tag{2.9}$$

The above relation is a thirteen term relation.. So, by defining $\mathcal{B}_6(F)$ which is factor subgroup defined as $Z[\mathbf{P}_F^1]/R_6(F)$, functional equations of $\mathcal{B}_6(F)$ is 13 term relation in which total number of terms are $13(11!) = 518918400$

Weight n. Goncharov defined generalized complex for $\mathcal{B}_n(F) = Z[\mathbf{P}_F^1]/R_n(F)$, where the subgroup $R_n(F)$ is kernel of the homomorphism $\delta_n : Z[\mathbf{P}_F^1] \to \mathcal{B}_{n-1}(F) \otimes F^\times$. Here, it is introduced as

$$r_n(\kappa_0, ..., \kappa_{2n-1}) = Alt_{2n} \frac{\triangle(\kappa_0, \kappa_1, ..., \kappa_n)\triangle(\kappa_0, \kappa_2, ..., \kappa_{n+1})...\triangle(\kappa_0, ..., \kappa_{2n-1})}{\triangle(\kappa_0, \kappa_1, ..., \kappa_{n+1})\triangle(\kappa_0, \kappa_2, ..., \kappa_{n+2})...\triangle(\kappa_0, ..., \kappa_n)} \tag{2.10}$$

Now let's define

$$R_n(F) = \sum_{i=0}^{2n} (-1)^i r_n(\kappa_0, ..., \hat{\kappa}_i, ..., \kappa_{2n}) \tag{2.11}$$

It is $2n + 1$ terms relation of the cross ratio of $2n$ points as $\mathcal{B}_n(F)$ is factor group of $Z[\mathbf{P}_F^1]/R_n(F)$ and $R_n(F)$, therefore the generalized functional equations of $\mathcal{B}_n(F)$ are $(2n + 1)(2n - 1)!$ terms for $n \geq 3$

$Weight$	Group	Defining Functional Equation
1	$\mathcal{B}_1(F)$	3 terms relation= 3 terms
2	$\mathcal{B}_2(F)$	5 terms relations= 5 terms
3	$\mathcal{B}_3(F)$	7 terms relations= 7(5)!= 840 terms
4	$\mathcal{B}_4(F)$	9 terms relations= 9(7)!= 45360 terms
5	$\mathcal{B}_5(F)$	11 terms relation=11(9)!= 3991680 terms
6	$\mathcal{B}_6(F)$	13 terms relation=13(11)!= 518918400 terms
$\vdots$	$\vdots$	$\vdots$
n	$\mathcal{B}_n(F)$	(2n+1) terms relations= (2n+1)(2n-1)! terms

3. Bi-complex Relation between Grassmannian and Classical Polylogarithmic Groups.

Goncharov proved that complexes between Grassmannian and classical polylogarithmic are bi-complex up to weight 3 given as

3.1. Weight 2.

$$\begin{tikzcd}
G_6(3) \arrow[r, "d"] \arrow[d, "p"] & G_5(3) \arrow[r, "d"] \arrow[d, "p"] & G_4(3) \arrow[d, "p"] \\
G_5(2) \arrow[r, "d"] & G_4(2) \arrow[r, "d"] \arrow[d, "f_1^2"] & G_3(2) \\
& \mathcal{B}_2(F) &
\end{tikzcd} \tag{B}$$

where

$$f_1^2(\kappa_0,\kappa_1,\kappa_2,\kappa_3) = \left[\frac{\triangle(\kappa_0,\kappa_3)\triangle(\kappa_1,\kappa_2)}{\triangle(\kappa_0,\kappa_2)\triangle(\kappa_1,\kappa_3)}\right]_2 \tag{3.12}$$

Lemma 3.2. *The generalized diagram B is a bi complex, therefore*

$$f_1^2 \circ p = 0 \tag{3.13}$$

Proof. For proof see [13] □

3.3. Weight 3.

$$\begin{tikzcd}
G_8(4) \arrow[r, "d"] \arrow[d, "p"] & G_7(4) \arrow[r, "d"] \arrow[d, "p"] & G_6(4) \arrow[d, "p"] \\
G_7(3) \arrow[r, "d"] & G_6(3) \arrow[r, "d"] \arrow[d, "f_2^3"] & G_5(3) \\
& \mathcal{B}_3(F) &
\end{tikzcd} \tag{C}$$

$$f_2^3(\kappa_0,...\kappa_5)=\frac{1}{15}Alt_6\Big[\frac{\triangle(\kappa_0,\kappa_1,\kappa_3)\triangle(\kappa_1,\kappa_2,\kappa_4)\triangle(\kappa_2,\kappa_0,\kappa_5)}{\triangle(\kappa_0,\kappa_1,\kappa_4)\triangle(\kappa_1,\kappa_2,\kappa_5)\triangle(\kappa_2,\kappa_0,\kappa_3)}\Big]_3 \tag{3.14}$$

Lemma 3.4. *The generalized diagram C is a bi complex, therefore*

$$f_2^3\circ p=0 \tag{3.15}$$

Proof. For proof, see [13] □

3.5. **Weight 4.** Now, by connecting Grassmannian complex and Bloch group in weight 4 it is given as

$$\begin{tikzcd} G_{10}(5) \arrow[r, "d"] \arrow[d, "p"] & G_9(5) \arrow[r, "d"] \arrow[d, "p"] & G_8(5) \arrow[d, "p"] \\ G_9(4) \arrow[r, "d"] & G_8(4) \arrow[r, "d"] \arrow[d, "f_3^4"] & G_7(4) \\ & \mathcal{B}_4(F) & \end{tikzcd} \tag{D}$$

$$f_3^4(\kappa_0,...\kappa_7)=\frac{1}{56}Alt_8\Big[\frac{\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_4)\triangle(\kappa_1,\kappa_2,\kappa_3,\kappa_5)...\triangle(\kappa_0,\kappa_1,\kappa_3,\kappa_7)}{\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_5)\triangle(\kappa_1,\kappa_2,\kappa_3,\kappa_6)...\triangle(\kappa_0,\kappa_1,\kappa_3,\kappa_4)}\Big]_4 \tag{3.16}$$

Lemma 3.6. *The generalized diagram D is a bi complex, therefore*

$$f_3^4\circ p=0 \tag{3.17}$$

Proof. let take nine points $(\kappa_0,...,\kappa_8)\in G_9(5)$, now use morphism p then

$$p(\kappa_0,...,\kappa_8)=\sum_{i=0}^{8}(-1)^i(\kappa_0,...,\hat{\kappa}_i,...\kappa_8) \tag{3.18}$$

then

$$f_3^4\circ p=\sum_{i=0}^{8}(-1)^i\frac{1}{56}Alt_8\Big[\frac{\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_4)\triangle(\kappa_1,\kappa_2,\kappa_3,\kappa_5)...\triangle(\kappa_0,\kappa_1,\kappa_3,\kappa_7)}{\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_5)\triangle(\kappa_1,\kappa_2,\kappa_3,\kappa_6)...\triangle(\kappa_0,\kappa_1,\kappa_3,\kappa_4)}\Big]_4 \tag{3.19}$$

It is 9 terms relation belongs to $\mathcal{B}_4(F)$ and equal to zero, therefore $f_3^4\circ p=0$ □

3.7. **Weight 5.**

$$\begin{tikzcd} G_{12}(6) \arrow[r, "d"] \arrow[d, "p"] & G_{11}(6) \arrow[r, "d"] \arrow[d, "p"] & G_{10}(6) \arrow[d, "p"] \\ G_{11}(45) \arrow[r, "d"] & G_{10}(5) \arrow[r, "d"] \arrow[d, "f_4^5"] & G_9(5) \\ & \mathcal{B}_5(F) & \end{tikzcd} \tag{E}$$

$$f_4^5(\kappa_0,...,\kappa_9)=\frac{1}{210}Alt_{10}\Big[\frac{\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_3,\kappa_5)...\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_4,\kappa_9)}{\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_3,\kappa_6)...\triangle(\kappa_0,\kappa_1,\kappa_2,\kappa_4,\kappa_5)}\Big]_5 \tag{3.20}$$

Lemma 3.8. *The generalized diagram* E *is a bi complex, therefore*

$$f_4^5 \circ p = 0 \tag{3.21}$$

Proof. let take eleven points $(\kappa_0, ..., \kappa_{10}) \in G_{11}(6)$, then

$$p(\kappa_0, ..., \kappa_{10}) = \sum_{i=0}^{10} (-1)^i (\kappa_0, ..., \hat{\kappa}_i, ...\kappa_{10}) \tag{3.22}$$

then

$$f_4^5 \circ p = \sum_{i=0}^{10} (-1)^i \frac{1}{210} Alt_{10} \Big[\frac{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_5)...\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_4, \kappa_9)}{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_6)...\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_4, \kappa_5)} \Big]_5 \tag{3.23}$$

It is 11 terms relation functional equation $\in \mathcal{B}_5(F)$ and equal to zero, therefore $f_4^5 \circ p = 0$ □

3.9. **Weight 6.**

$$\begin{tikzcd} G_{14}(7) \arrow[r, "d"] \arrow[d, "p"] & G_{13}(7) \arrow[r, "d"] \arrow[d, "p"] & G_{12}(7) \arrow[d, "p"] \\ G_{13}(6) \arrow[r, "d"] & G_{12}(6) \arrow[r, "d"] \arrow[d, "f_5^6"] & G_{11}(6) \\ & \mathcal{B}_6(F) & \end{tikzcd} \tag{F}$$

$$f_5^6(\kappa_0, ..., \kappa_{11}) = \frac{1}{792} Alt_{12} \Big[\frac{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_6)...\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_5, \kappa_{11}}{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_7)...\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_5, \kappa_6)} \Big]_6 \tag{3.24}$$

Lemma 3.10. *The generalized diagram* F *is a bi complex, therefore*

$$f_5^6 \circ p = 0 \tag{3.25}$$

Proof. let $(\kappa_0, ..., \kappa_{12}) \in G_{13}(7)$, then

$$p(\kappa_0, ..., \kappa_{12}) = \sum_{i=0}^{12} (-1)^i (\kappa_0, ..., \hat{\kappa}_i, ...\kappa_{12}) \tag{3.26}$$

$$f_5^6 \circ p = \sum_{i=0}^{12} (-1)^i \frac{1}{792} Alt_{12} \Big[\frac{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_6)...\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_5, \kappa_{11})}{\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_7)...\triangle(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_5, \kappa_6)} \Big]_6 \tag{3.27}$$

It is 13 terms relation $\in \mathcal{B}_6(F)$ and equal to zero. So $f_5^6 \circ p = 0$ □

3.11. Weight n.

$$\begin{tikzcd} G_{2n+2}(n+1) \arrow[r, "d"] \arrow[d, "p"] & G_{2n+1}(n+1) \arrow[r, "d"] \arrow[d, "p"] & G_{2n}(n+1) \arrow[d, "p"] \\ G_{2n+1}(n) \arrow[r, "d"] & G_{2n}(n) \arrow[r, "d"] \arrow[d, "f^n_{n-1}"] & G_{2n-1}(n) \\ & \mathcal{B}_n(F) & \end{tikzcd} \tag{G}$$

$$f^n_{n-1} = \frac{1}{{}^{2n}C_{n+1}} Alt_{2n}\Big[\frac{\triangle(\kappa_0,\kappa_1,...,\kappa_n)\triangle(\kappa_0,\kappa_2,...,\kappa_{n+1})...\triangle(\kappa_0,...,\kappa_{2n-1})}{\triangle(\kappa_0,\kappa_1,...,\kappa_{n+1})\triangle(\kappa_0,\kappa_2,...,\kappa_{n+2})...\triangle(\kappa_0,...,\kappa_n)}\Big]_n \tag{3.28}$$

where $n \geq 3$

Theorem 3.12. *The generalized diagram G is a bi complex therefore*

$$f^n_{n-1} \circ p = 0 \tag{3.29}$$

Proof. let $(\kappa_0,...,\kappa_{2n}) \in G_{2n+1}(n+1)$ apply map p we get

$$p(\kappa_0,...,\kappa_{2n}) = \sum_{i=0}^{2n}(-1)^i(\kappa_0,...,\hat{\kappa}_i,...\kappa_{2n}) \tag{3.30}$$

$$f^n_{n-1} \circ p = \sum_{i=0}^{2n}(-1)^i \frac{1}{{}^{2n}C_{n+1}} Alt_{2n}\Big[\frac{\triangle(\kappa_0,\kappa_1,...,\kappa_n)\triangle(\kappa_0,\kappa_2,...,\kappa_{n+1})...\triangle(\kappa_0,...,\kappa_{2n-1})}{\triangle(\kappa_0,\kappa_1,...,\kappa_{n+1})\triangle(\kappa_0,\kappa_2,...,\kappa_{n+2})...\triangle(\kappa_0,...,\kappa_n)}\Big]_n \tag{3.31}$$

It is $(2n+1)$ terms relation $\in \mathcal{B}_n(F)$ and equal to zero, therefore $f^n_{n-1} \circ p = 0$ □

4. Conclusion

In this research work, generalized functional equations of classical polylogarithmic group $\mathcal{B}_n(F)$ are introduced. Upto weight $n = 4$, the functional equations have been derived by other researchers, in this work it is extended upto any weight $n \in N$.
This work has also generalized bi-complex form of Grassmannian configuration and classical polylogarithmic group chain complexes. Classical polylogarithmic groups have many differential forms like tangential and infinitesimal therefore these generalized functional equations will help other researchers to introduce functional equations of differential form of polylogarithmic groups.

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