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Hermite-Hadamard Type Inequality And Some Related Results For Mittag-Leffler Type Convex Functions With Applications

Sabir Hussain[1], Sobia Rafeeq[2*]
Department of Mathematics,
University of Engineering and Technology, Lahore, Pakistan,
Department of Mathematics,
MNS University of Engineering and Technology, Multan, Pakistan,
Email: sabirhus@gmail.com[1]; sobia83zartash@hotmail.com[2*]
*Corresponding author sobia83zartash@hotmail.com

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Abstract.: Here, a multiparameters fractional integral identity has been proved to provide some new estimates for Hermite-Hadamard type, trapezoidal type, midpoint type and Simpson type functionals in the context of Mittag-Leffler type convex functions to provide some applications.

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1. Introduction

Fractional calculus is the study of integrals and derivatives of arbitrary order which was a natural outgrowth of conventional definitions of calculus integral and derivative. There are several problems in the mathematics and its related real world applications wherein fractional derivatives occupy an important place [1, 2, 13, 15, 18, 19, 26, 27]. Each conventional fractional operator with its own special kernel can be used in a certain problem. In recent years, fractional calculus is a topic that has played a crucial role in defining the complex dynamics of the real world problems from various fields of science and engineering. Analyzing the uniqueness of solution of fractional ordinary and partial differential equations can be performed by employing fractional integral inequalities [7, 8, 9, 20, 21, 28]. While, on the other hand the role of elementary mathematical inequalities have been rediscovered owing to their applications to different realms of mathematics and applied sciences [22]. In fact, the development of mathematical inequalities is very closely related to the advances in the theory of convex function. Convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics. As the development of the convex function is associated to many well known names of the era, for instance, Jensen, Hardy, Ostrowski, Hadamard etc. [5, 6, 14, 25]. One of the most celebrated and sparkled results on convex function is due to Hermite-Hadamard, later on called Hermite-Hadamard integral inequality. Due to its geometrical significance, these inequalities were either generalized, extended and refined by using elementary techniques of analysis. The following inequality is well known as Simpson's inequality which provides an error bound for the Simpson's rule.

Theorem 1.1. [12] *Let $f : [a_1, a_2] \to \mathbb{R}$ be a four times differentiable function on (a_1, a_2) and $\|f^{(4)}\|_\infty := \sup_{x\in(a_1,a_2)} |f^{(4)}|(x) < \infty$, then the following inequality holds*

$$\left|\frac{1}{3}\left[\frac{f(a_1)+f(a_2)}{2} + 2f\left(\frac{a_1+a_2}{2}\right)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)dx\right| \le \frac{\|f^{(4)}\|_\infty (a_2-a_1)^4}{2880}.$$

The classical Simpson type inequality has attracted much attention since it is very remarkable in the area of inequality application. For different convex functions, the Simpson's inequality have been extended and refined by many authors such as [10, 4, 16]. Dragomir et al. proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

Theorem 1.2. [10] *Suppose $f : [a_1, a_2] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a_1, a_2) and $f' \in L[a_1, a_2]$, then*

$$\left|\frac{1}{3}\left[\frac{f(a_1)+f(a_2)}{2} + 2f\left(\frac{a_1+a_2}{2}\right)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)dx\right| \le \frac{a_2-a_1}{3}||f'||_1$$

provided that: $\|f'\|_1 = \int_{a_1}^{a_2} |f'(x)|dx < \infty$.

The aim of this paper is to obtain a fractional integral identity for Mittag-Leffler type convex function, consequently to derive some new Hermite-Hadamard type, trapezoidal type, mid point type and Simpson type fractional integral inequalities to provide some applications involving special means. This paper is organized in the following way. After this Introduction, in Section 2 some basic concepts and assumptions are discussed. In Section 3 some results relating to the topic are established, in Section 4 some applications of the obtained results are given. Lastly, Section 5 is about conclusion of the current paper.

2. Preliminaries and Assumptions

Definition 2.1. [15] *The single-parameter Mittag-Leffler function and the two parameter Mittag-Leffler function are defined, respectively, as:*

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+1)}; \;\; E_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\beta)}, \;\; \alpha, \beta > 0. \tag{2. 1}$$

Definition 2.2. *For $\alpha > 0$, the logarithmic Mittag-Leffler mean of a given function $f(x)$ on $[a_1, a_2]$ is defined as:*

$$LME(x) := \ln \frac{E_\alpha(f(x)) + E_\alpha(f(a_1+a_2-x))}{2}, \;\; x \in [a_1, a_2]. \tag{2. 2}$$

If f is differentiable on (a_1, a_2), then

$$LME'(x) := \frac{E_{\alpha,\alpha}(f(x))f'(x) - E_{\alpha,\alpha}(f(a_1+a_2-x))f'(a_1+a_2-x)}{\alpha[E_\alpha(f(x)) + E_\alpha(f(a_1+a_2-x))]}. \tag{2. 3}$$

For $\alpha \to 1$, relations (2.2) and (2.3) will be degenerated to the following relations [17]

$$LE(x) := \ln \frac{\exp(f(x)) + \exp(f(a_1 + a_2 - x))}{2}, \quad x \in [a_1, a_2], \tag{2. 4}$$

$$LE'(x) := \frac{\exp(f(x))f'(x) - \exp(f(a_1 + a_2 - x))f'(a_1 + a_2 - x)}{\exp(f(x)) + \exp(f(a_1 + a_2 - x))]}. \tag{2. 5}$$

Definition 2.3. [17] *A function $f : [a_1, a_2] \subseteq \mathbf{R} \to \mathbf{R}$, is said to be single and double parameter Mittag-Leffler type convex function, respectively, if the following inequalities hold:*

$$E_\alpha(f(tx+(1-t)y)) \le tE_\alpha(f(x))+(1-t)E_\alpha(f(y)),\ t \in [0,1];\ x, y \in [a_1, a_2], \tag{2. 6}$$

$$E_{\alpha,\beta}(f(tx + (1-t)y)) \le tE_{\alpha,\beta}(f(x)) + (1-t)E_{\alpha,\beta}(f(y)), t\in [0,1]; x, y \in [a_1, a_2]. \tag{2. 7}$$

It is remarkable to note that for $\alpha, \beta \to 1$, the Mittag-Leffler type convex functions defined by Definition 2.3 will degenerate into classic exp-convex function consistently [17].

Definition 2.4. [11] *Let $I \subseteq (0, \infty)$ be a real interval and $0 \neq p \in \mathbf{R}$. A function $f : I \to \mathbf{R}$ is said to be $p-$convex function, if*

$$f((tx^p + (1-t)y^p)^{\frac{1}{p}}) \le tf(x) + (1-t)f(y),$$

provided that $x, y \in I$ and $t \in [0, 1]$. If the inequality is reversed, the f is said to be $p-$concave function. The function f is said to be $(s, p)-$convex if

$$f((tx^p + (1-t)y^p)^{\frac{1}{p}}) \le t^s f(x) + (1-t)^s f(y),$$

provided that $x, y \in I$; $t \in [0, 1]$ and $s \in (0, 1]$. In case of $(s, p)-$concave function, the inequality sign is reversed.

Definition 2.5. [24] *Let $[a_1, a_2]$ be a finite interval on the real axis and $f \in L[a_1, a_2]$. The right-hand side and the left-hand side Riemann-Liouville fractional integrals $\mathcal{J}^\alpha_{a_1+}f$ and $\mathcal{J}^\alpha_{a_2-}f$ of order $\alpha > 0$, respectively, are defined by:*

$$\left(\mathcal{J}^\alpha_{a_1+}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} (x-t)^{\alpha-1} f(t)dt,\ x \in (a_1, a_2], \tag{2. 8}$$

$$\left(\mathcal{J}^\alpha_{a_2-}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{a_2} (t-x)^{\alpha-1} f(t)dt,\ x \in [a_1, a_2), \tag{2. 9}$$

provided that $L[a_1, a_2]$ is the space of all Lebesgue integrable functions on the compact interval $[a_1, a_2]$.

Definition 2.6. [24] *The gamma function, Γ, beta function, $\mathbb{B}$ and the Hypergeometric function, ${}_2F_1$, respectively, defined by:*

$$\Gamma(x) := \int_0^\infty e^{-t} t^x dt,\ x > 0;\ \ \mathbb{B}(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x, y > 0.$$

$${}_2F_1(a_1, a_2; c, z) := \frac{1}{\mathbb{B}(a_2, c-a_2)} \int_0^1 t^{a_2-1}(1-t)^{c-a_2-1}(1-zt)^{-a_1}dt,\ c > a_2 > 0; |z| < 1.$$

Raina [23] introduced a class of functions defined by:

$$\mathfrak{F}^{\sigma}_{\rho,\lambda}(x) = \mathfrak{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \quad \rho, \lambda \in \mathbf{R}^+; x \in \mathbf{R}, \tag{2. 10}$$

where the coefficients $\sigma(k) \in \mathbf{R}^+$, $k \in \mathbb{N}_0$ form a bounded sequence. By using (2.10) Agarwal et al. and Raina [3, 23] defined, respectively, the left-side and right-sided fractional integral operators:

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,a_1+;w}\phi\right)(x) = \int_{a_1}^{x} (x-t)^{\lambda-1} \mathfrak{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\phi(t)dt, \; x > a_1. \tag{2. 11}$$

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,a_2-;w}\phi\right)(x) = \int_{x}^{a_2} (t-x)^{\lambda-1} \mathfrak{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\phi(t)dt, \; x < a_2, \tag{2. 12}$$

where $w \in \mathbf{R}$ and ϕ is a function such that the integrals on right hand sides exit. It is easy to verify that $\mathfrak{J}^{\sigma}_{\rho,\lambda,a_1+;w}\phi(x)$ and $\mathfrak{J}^{\sigma}_{\rho,\lambda,a_2-;w}\phi(x)$ are bounded integral operators on $L(a_1, a_2)$, provided that $\mathfrak{M} := \mathfrak{F}^{\sigma}_{\rho,\lambda+1}[w(a_2 - a_1)^{\rho}] < \infty$. In fact, for $\phi \in L(a_1, a_2)$, we have

$$\left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,a_1+;w}\phi\right\|_1 \leq \mathfrak{M}(a_2 - a_1)^{\lambda}\|\phi\|_1; \quad \left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,a_2-;w}\phi\right\|_1 \leq \mathfrak{M}(a_2 - a_1)^{\lambda}\|\phi\|_1.$$

By setting $\lambda \to \alpha$; $\sigma(0) \to 1$ and $w \to 0$ in (2.11) and (2.12), respectively, (2.8) and (2.9) are recaptured. Before starting the main results, we consider the following consider the following notations to make the representation easier and more compact.

$$\begin{aligned}
\Psi^{\mu}_{\lambda}(a_1, a_2, p, \kappa; \alpha) := {} & \frac{p(\lambda+\mu+\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p - a_1^p)^{\rho}]-1)}{2(1-\lambda_1)} \ln \frac{E_\alpha(f(a_1))+E_\alpha(f(a_2))}{2} + \\
& \frac{p}{2(1-\lambda_1)}\{(1-\lambda)(LME\circ g)\left(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)\right) - \\
& (\mu - \mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p - a_1^p)^{\rho}])(LME\circ g)\left(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)\right)\} - \\
& p\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\beta,[(1-\lambda_1)(a_1^p-\kappa)+a_2^p]+;\frac{w(a_2^p-a_1^p)^{\rho}}{(1-\lambda_1)^{\rho}(\kappa-a_1^p)^{\rho}}} LME\circ g\right)(a_2^p)}{2(1-\lambda_1)^{1+\beta}(\kappa - a_1^p)^{\beta}} + \\
& p\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\beta,a_1^p+;\frac{w(a_2^p-a_1^p)^{\rho}}{(1-\lambda_1)^{\rho}(a_2^p-\kappa)^{\rho}}} LME\circ g\right)((1-\lambda_1)(a_2^p-\kappa)+a_1^p)}{2(1-\lambda_1)^{1+\beta}(a_2^p-\kappa)^{\beta}}.
\end{aligned} \tag{2. 13}$$

For different choices of parameters in (2.13), we have some well known functionals for Mittag-Leffler type convex functions. For instance, $\lambda, \mu, \beta, p, \sigma(0) \to 1$; $w, \lambda_1 \to 0$; $\kappa \to \frac{a_1^p + a_2^p}{2}$, relation (2.13) reduces to following trapezoidal type functional

$$\Psi^{1}_{1}\left(a_1, a_2, 1, \frac{a_1+a_2}{2}; \alpha\right) := \ln \frac{E_\alpha(f(a_1))+E_\alpha(f(a_2))}{2} - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} LME(t)dt. \tag{2. 14}$$

For $\beta, p, \sigma(0) \to 1; \lambda, \mu, w, \lambda_1 \to 0; \kappa \to \frac{a^p+a_2^p}{2}$, relation (2.13) reduces to following mid point type functional

$$\Psi_0^0\left(a_1, a_2, 1, \frac{a_1+a_2}{2}; \alpha\right) := LME\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} LME(t)dt. \quad (2.\ 15)$$

For $\beta, p, \sigma(0) \to 1; w, \lambda_1 \to 0; \kappa \to \frac{a_1^p+a_2^p}{2}, \lambda, \mu \to \frac{1}{3}$ relation (2.13) reduces to following Simpson's type functional

$$\Psi_{\frac{1}{3}}^{\frac{1}{3}}\left(a_1, a_2, 1, \frac{a_1+a_2}{2}; \alpha\right) := \frac{1}{3}\left[\ln\frac{E_\alpha(f(a_1))+E_\alpha(f(a_2))}{2} + 2LME\left(\frac{a_1+a_2}{2}\right)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} LME(t)dt \quad (2.\ 16)$$

$$G_1(a_1, a_2, s, p, \lambda, \mu, \kappa) := \frac{|1-\lambda|(a_2^p-\kappa)}{2(s+1)\left[a_1^p+(1-\lambda_1)(a_2^p-\kappa)\right]^{\frac{p-1}{p}}}$$
$$\{|LME'(a_1)|\mathfrak{G}_s^s(a_1, a_2, p; 1; \eta_4; 0) + (|LME'(a_1)|\lambda_1^s + (1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|)$$
$$\mathfrak{G}_s^0(a_1, a_2, p; 1; \eta_4; 0)\} + \frac{|\mu|(\kappa-a_1^p)}{2(s+1)a_2^{p-1}}\{|LME'(a_2)|\mathfrak{G}_s^0(a_1, a_2, p; 1; 0; \eta_5)$$
$$+ (|LME'(a_2)|\lambda_1^s + (1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|)\mathfrak{G}_s^s(a_1, a_2, p; 1; 0; \eta_5)\} \quad (2.\ 17)$$

$$\sigma_1(k) := \sigma(k)\left[\frac{a_2^p-\kappa}{2\left[a_1^p+(1-\lambda_1)(a_2^p-\kappa)\right]^{\frac{p-1}{p}}}\left\{|LME'(a_1)|\mathfrak{H}_0^{1,s}(a_1, a_2, k, p; \eta_4; 0)\right.\right.$$
$$+ (|LME'(a_1)|\lambda_1^s + (1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|)\mathfrak{L}^s(a_1, a_2, k, p; \eta_4; 0)\}$$
$$+ \frac{\kappa-a_1^p}{2a_2^{p-1}} \times \{|LME'(a_2)|\mathfrak{L}^s(a_1, a_2, k, p; 0; \eta_5)$$
$$\left.\left.+(|LME'(a_2)|\lambda_1^s + (1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|)\mathfrak{H}_0^{1,s}(a_1, a_2, k, p; 0; \eta_5)\right\}\right] \quad (2.\ 18)$$

$$\sigma_2(k) := \left\{\frac{(a_2^p-\kappa)\sqrt[y]{|LME'(a_1)|^y(1+\lambda_1^s) + (1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^y}}{2\sqrt[y]{s+1}\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x(p-1)}}\right.$$
$$\times \sqrt[x]{\mathfrak{H}_0^{x,0}(a_1, a_2, k, p; \eta_4; 0)} + \sqrt[x]{\mathfrak{H}_0^{x,0}(a_1, a_2, k, p; 0; \eta_5)} \times$$
$$\left.\frac{(\kappa-a_1^p)\sqrt[y]{|LME'(a_2)|^y(1+\lambda_1^s) + (1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^y}}{2\sqrt[y]{s+1}a_2^{x(p-1)}}\right\}\sigma(k) \quad (2.\ 19)$$

$$G_2(a_1,a_2,s,p,\lambda,\mu,\kappa) := \frac{|1-\lambda|(a_2^p-\kappa)\sqrt[x]{\mathfrak{G}_0^0(a_1,a_2,p;x;\eta_4;0)}}{2\sqrt[y]{s+1}\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x(p-1)}}$$
$$\times \sqrt[y]{|LME'(a_1)|^y(1+\lambda_1^s)+(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^y}$$
$$+\frac{|\mu|(\kappa-a_1^p)\sqrt[x]{\mathfrak{G}_0^0(a_1,a_2,p;x;0;\eta_5)}}{2\sqrt[y]{s+1}a_2^{x(p-1)}}$$
$$\times \sqrt[y]{|LME'(a_2)|^y(1+\lambda_1^s)+(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^y} \quad (2.\ 20)$$

$$\sigma_3(k) := \sigma(k)\left[\frac{(a_2^p-\kappa)\{\mathfrak{H}_0^{1,0}(a_1,a_2,k,p;\eta_4;0)\}^{\frac{x-1}{x}}}{2\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{p-1}}\{\mathfrak{H}_0^{1,s}(a_1,a_2,k,p;\eta_4;0)\right.$$
$$\times|LME'(a_1)|^x+\{\lambda_1^s|LME'(a_1)|^x+(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^x\}$$
$$\mathfrak{H}_s^{1,0}(a_1,a_2,k,p;0;\eta_5)\}^{\frac{1}{x}}+\frac{(\kappa-a_1^p)\{\mathfrak{H}_0^{1,0}(a_1,a_2,k,p;0;\eta_5)\}^{\frac{x-1}{x}}}{2a_2^{p-1}}$$
$$\times\{\mathfrak{L}^s(a_1,a_2,k,p;0;\eta_5)|LME'(a_2)|^x+\{\lambda_1^s|LME'(a_2)|^x$$
$$\left.+(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^x\}\mathfrak{H}_0^{1,s}(a_1,a_2,k,p;0;\eta_5)\}^{\frac{1}{x}}\right] \quad (2.\ 21)$$

$$G_3(a_1,a_2,s,p,\lambda,\mu,\kappa) := \frac{|1-\lambda|(a_2^p-\kappa)\{\mathfrak{G}_0^0(a_1,a_2,p;1;\eta_4;0)\}^{\frac{x-1}{x}}}{2\sqrt[x]{s+1}\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{p-1}}$$
$$\{\mathfrak{G}_s^s(a_1,a_2,p;1;\eta_4;0)|LME'(a_1)|^x+\{\lambda_1^s|LME'(a_1)|^x$$
$$+(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^x\}\mathfrak{G}_s^0(a_1,a_2,p;1;\eta_4;0)\}^{\frac{1}{x}}$$
$$+\frac{(\kappa-a_1^p)|\mu|\{\mathfrak{G}_0^0(a_1,a_2,p;1;0;\eta_5)\}^{\frac{x-1}{x}}}{2a_2^{p-1}\sqrt[x]{s+1}}\{\mathfrak{G}_s^0(a_1,a_2,p;1;0;\eta_5)|LME'(a_2)|^x$$
$$+\{\lambda_1^s|LME'(a_2)|^x+(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^x\}$$
$$\times\mathfrak{G}_s^s(a_1,a_2,p;1;0;\eta_5)\}^{\frac{1}{x}} \quad (2.\ 22)$$

$$\mathfrak{H}_{\eta_3}^{\eta_1,\eta_2}(a_1,a_2,k,p;\eta_4;\eta_5) := \mathbb{B}(\eta_1\beta+\eta_1k\rho+\eta_2+1,1+\eta_3)\times$$
$${}_2F_1\left(\eta_1\frac{p-1}{p},\eta_1\beta+\eta_1k\rho+\eta_2+\eta_3+1;\eta_1\beta+\eta_1k\rho+\eta_2+\eta_3+2,\eta_4+\eta_5\right) \quad (2.\ 23)$$

$$\mathfrak{G}_{\eta_3}^{\eta_2}(a_1,a_2,p;\eta_1;\eta_4;\eta_5) := {}_2F_1\left(\eta_1\frac{p-1}{p},\eta_2+1;\eta_3+2,\eta_4+\eta_5\right) \quad (2.\ 24)$$

$$\mathfrak{L}^{\eta_2}(a_1,a_2,k,p;\eta_4;\eta_5) := \mathbb{B}(\beta+k\rho+1,1+\eta_2)\times {}_2F_1\left(\frac{p-1}{p},\beta+k\rho+1;\beta+k\rho+\eta_2+2,\eta_4+\eta_5\right) \quad (2.\ 25)$$

$$\eta_4 := \frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}; \quad \eta_5 := \frac{(1-\lambda_1)(\kappa-a_1^p)}{a_2^p}. \quad (2.\ 26)$$

3. Results

Lemma 3.1. *Let* $I_p := [a_1^p, a_2^p] \subseteq \mathbf{R}_+$ *for* $0 \neq p \in \mathbf{R}$ *with* $a_1 < a_2$ *and* I_p° *an interior of* I_p*; let* $f : I_p \to \mathbf{R}$ *be a differentiable function on* I_p°*. Moreover, if* $\lambda, \mu \in \mathbf{R}$, $\kappa \in I_p$, $\lambda_1 \in [0,1]$, $\rho, \beta, \xi > 0$ *and* $g(\xi) = \sqrt[p]{\xi}$*, then*

$$\begin{aligned}
&\frac{a_2^p-\kappa}{2}\int_0^1 \{1-\lambda-t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho t^\rho]\} \\
&\times LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right) \\
&\times[ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt+\frac{\kappa-a_1^p}{2}\int_0^1(\mu-t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho t^\rho]) \\
&\times LME'(\sqrt[p]{(1-t)a_2^p+t(\lambda_1 a_2^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}) \\
&\times[(1-t)a_2^p+t(\lambda_1 a_2^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt \\
&= \frac{p(\lambda+\mu+\mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho]-1)}{2(1-\lambda_1)}\ln\frac{E_\alpha(f(a_1))+E_\alpha(f(a_2))}{2}+ \\
&\frac{p}{2(1-\lambda_1)}\{(1-\lambda)(LME\circ g)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))- \\
&(\mu-\mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho])(LME\circ g)(\lambda_1 a_2^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))\}- \\
&p\frac{\left(\mathfrak{J}^\sigma_{\rho,\beta,[(1-\lambda_1)(a_1^p-\kappa)+a_2^p]+;\frac{w(a_2^p-a_1^p)^\rho}{(1-\lambda_1)^\rho(\kappa-a_1^p)^\rho}}LME\circ g\right)(a_2^p)}{2(1-\lambda_1)^{1+\beta}(\kappa-a_1^p)^\beta}+ \\
&p\frac{\left(\mathfrak{J}^\sigma_{\rho,\beta,a_1^p+;\frac{w(a_2^p-a_1^p)^\rho}{(1-\lambda_1)^\rho(a_2^p-\kappa)^\rho}}LME\circ g\right)((1-\lambda_1)(a_2^p-\kappa)+a_1^p)}{2(1-\lambda_1)^{1+\beta}(a_2^p-\kappa)^\beta}. \quad (3.\ 27)
\end{aligned}$$

Proof. Integrating by parts and changing variable yields the following:

$$\begin{aligned}
I_1 &:= \int_0^1 \{1-\lambda-t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho t^\rho]\} \\
&\times LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right) \\
&\times[ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt
\end{aligned}$$

$$
\begin{aligned}
&= \left| p \frac{(LME \circ g)\left(ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))\right)}{(1-\lambda_1)(\kappa - a_2^p)} \right. \\
&\times \{1 - \lambda - t^{\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p - a_1^p)^{\rho} t^{\rho}]\}\Big|_0^1 + p \int_0^1 t^{\beta-1} \mathfrak{F}^{\sigma}_{\rho,\beta}[w(a_2^p - a_1^p)^{\rho} t^{\rho}] \\
&\times \frac{(LME \circ g)\left(ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))\right)}{(1-\lambda_1)(\kappa - a_2^p)} dt \\
&= [(1-\lambda_1)(\kappa - a_2^p)]^{-1} \{p(1 - \lambda - \mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p - a_1^p)^{\rho}]) LME(a_1) \\
&+ (\lambda - 1) p (LME \circ g)\left(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)\right)\} \\
&+ \frac{p}{(1-\lambda_1)(\kappa - a_2^p)} \int_0^1 t^{\beta-1} \mathfrak{F}^{\sigma}_{\rho,\beta}[w(a_2^p - a_1^p)^{\rho} t^{\rho}] \\
&\times (LME \circ g)\left(ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))\right) dt.
\end{aligned}
$$

By setting, $x = ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))$ so that $\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa) - x = t(1-\lambda_1)(a_2^p - \kappa) \Rightarrow dt = -\frac{dx}{(1-\lambda_1)(a_2^p - \kappa)}$ and $0 \le t \le 1 \Leftrightarrow (1-\lambda_1)(a_2^p - \kappa) + a_1^p \le x \le a_1^p$

$$
\begin{aligned}
I_1 &= [(1-\lambda_1)(a_2^p - \kappa)]^{-1} \{p(\lambda + \mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p - a_1^p)^{\rho}] - 1) LME(a_1) \\
&+ (1-\lambda) p (LME \circ g)\left(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)\right)\} \\
&- \frac{p}{[(1-\lambda_1)(a_2^p - \kappa)]^{\beta+1}} \int_{a_1^p}^{(1-\lambda_1)(a_2^p - \kappa) + a_1^p} (a_1^p + (1-\lambda_1)(a_2^p - \kappa) - x)^{\beta-1} \\
&\times \mathfrak{F}^{\sigma}_{\rho,\beta} \left[w \left\{ \frac{(a_2^p - a_1^p)((1-\lambda_1)(a_2^p - \kappa) + a_1^p - x)}{(1-\lambda_1)(a_2^p - \kappa)} \right\}^{\rho} \right] (LME \circ g)(x) dx,
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \ (1-\lambda_1)(a_2^p - \kappa) I_1 &= p(\lambda + \mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p - a_1^p)^{\rho}] - 1) LME(a_1) \\
&+ (1-\lambda) p LME \left(\sqrt[p]{\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)} \right) \\
&- \frac{p}{[(1-\lambda_1)(a_2^p - \kappa)]^{\beta}} \int_{a_1^p}^{(1-\lambda_1)(a_2^p - \kappa) + a_1^p} (a_1^p + (1-\lambda_1)(a_2^p - \kappa) - x)^{\beta-1} \\
&\times \mathfrak{F}^{\sigma}_{\rho,\beta} \left[w \left\{ \frac{(a_2^p - a_1^p)((1-\lambda_1)(a_2^p - \kappa) + a_1^p - x)}{(1-\lambda_1)(a_2^p - \kappa)} \right\}^{\rho} \right] (LME \circ g)(x) dx.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
(1-\lambda_1)(a_2^p - \kappa) I_1 &= p(\lambda + \mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p - a_1^p)^{\rho}] - 1) LME(a_1) \\
&+ (1-\lambda) p (LME \circ g)\left(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)\right) - \frac{p}{[(1-\lambda_1)(a_2^p - \kappa)]^{\beta}} \\
&\times \left(\mathfrak{J}^{\sigma}_{\rho,\beta,a_1^p+;\frac{w(a_2^p - a_1^p)^{\rho}}{(1-\lambda_1)^{\rho}(a_2^p - \kappa)^{\rho}}} LME \circ g \right) ((1-\lambda_1)(a_2^p - \kappa) + a_1^p). \quad (3.\ 28)
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}
I_2 &:= \int_0^1 (\mu - t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho t^\rho]) \\
&\times LME'(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))})dt \\
&\times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \\
&= p[(1-\lambda_1)(a_1^p - \kappa)]^{-1} |(\mu - t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho t^\rho]) \\
&\times (LME \circ g)((1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)))|_0^1 + p\int_0^1 t^{\beta-1} \\
&\times \mathfrak{F}^\sigma_{\rho,\beta}[w(a_2^p - a_1^p)^\rho t^\rho] \frac{LME(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))})}{(1-\lambda_1)(a_1^p - \kappa)} dt \\
&= p[(1-\lambda_1)(\kappa - a_1^p)]^{-1} \{\mu LME(a_2) - (\mu - \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho]) \\
&\times (LME \circ g)(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))\} \\
&- \frac{p}{(1-\lambda_1)(\kappa - a_1^p)} \int_0^1 t^{\beta-1} \mathfrak{F}^\sigma_{\rho,\beta}[w(a_2^p - a_1^p)^\rho t^\rho] \\
&\times (LME \circ g)((1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)))dt.
\end{aligned}$$

By setting, $y = (1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))$ so that $a_2^p - y = t(1-\lambda_1)(\kappa - a_1^p)$ $\Rightarrow dt = \frac{dy}{(1-\lambda_1)(a_1^p - \kappa)}$ and $0 \le t \le 1 \Leftrightarrow (1-\lambda_1)(a_1^p - \kappa) + a_2^p \ge y \ge a_2^p$

$$\begin{aligned}
(1-\lambda_1)(\kappa - a_1^p) I_2 &= p[\mu LME(a_2) - (\mu - \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho]) \\
&\times (LME \circ g)(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))] - \frac{p}{[(1-\lambda_1)(\kappa - a_1^p)]^\beta} \\
\times \int_{(1-\lambda_1)(a_1^p - \kappa) + a_2^p}^{a_2^p} & (a_2^p - y)^{\beta-1} \mathfrak{F}^\sigma_{\rho,\beta} \left[w \left\{ \frac{(a_2^p - a_1^p)(a_2^p - y)}{(1-\lambda_1)(\kappa - a_1^p)} \right\}^\rho \right] (LME \circ g)(y) dy.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
(1-\lambda_1)(\kappa - a_1^p) I_2 &= p[\mu LME(a_2) - (\mu - \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho]) \\
&\times (LME \circ g)(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))] \\
&- \frac{p}{[(1-\lambda_1)(\kappa - a_1^p)]^\beta} \left(\mathfrak{J}^\sigma_{\rho,\beta,[(1-\lambda_1)(a_1^p - \kappa) + a_2^p]+;\frac{w(a_2^p - a_1^p)^\rho}{(1-\lambda_1)^\rho(\kappa - a_1^p)^\rho}} LME \circ g \right)(a_2^p).
\end{aligned} \tag{3. 29}$$

Addition of (3.28) and (3.29), yields the following:

$$(1-\lambda_1)[(a_2^p-\kappa)I_1+(\kappa-a_1^p)I_2]=p(\lambda+\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho]-1)LME(a_1)$$
$$+(1-\lambda)p(LME\circ g)\left(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa)\right)+p\mu LME(a_2)$$
$$-p(\mu-\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho])(LME\circ g)\left(\lambda_1 a_2^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa)\right)]$$
$$-\frac{p}{[(1-\lambda_1)(\kappa-a_1^p)]^\beta}\left(\mathfrak{J}^{\sigma}_{\rho,\beta,[(1-\lambda_1)(a_1^p-\kappa)+a_2^p]+;\frac{w(a_2^p-a_1^p)^\rho}{(1-\lambda_1)^\rho(\kappa-a_1^p)^\rho}}LME\circ g\right)(a_2^p)$$
$$-\frac{p}{[(1-\lambda_1)(a_2^p-\kappa)]^\beta}\left(\mathfrak{J}^{\sigma}_{\rho,\beta,a_1^p+;\frac{w(a_2^p-a_1^p)^\rho}{(1-\lambda_1)^\rho(a_2^p-\kappa)^\rho}}LME\circ g\right)((1-\lambda_1)(a_2^p-\kappa)+a_1^p).$$

This completes the proof. □

Theorem 3.2. *Let $f: I_p\subseteq\mathbf{R}^+\to\mathbf{R}$ be a differentiable function on I_p°, interior of I_p, such that $|LME'|$ is $(s,p)-$convex for $0\neq p\in\mathbf{R}$ and $\rho,\beta>0$; let $g(\xi)=\sqrt[p]{\xi}$, $\xi>0$ and $(s,\lambda_1)\in(0,1]\times[0,1]$, then*

$$|\Psi^{\mu}_{\lambda}(a_1,a_2,p,\kappa;\alpha)|\leq\mathfrak{F}^{\sigma_1}_{\rho,\beta+1}[|w|(a_2^p-a_1^p)^\rho]+G_1(a_1,a_2,s,p,\lambda,\mu,\kappa). \tag{3. 30}$$

Proof. By relation (2.13), using the property of absolute, the following holds:

$$|\Psi^{\mu}_{\lambda}(a_1,a_2,p,\kappa;\alpha)|\leq\frac{(a_2^p-\kappa)|I_1|+(\kappa-a_1^p)|I_2|}{2}. \tag{3. 31}$$

By $(s,p)-$convexity of $|LME'|$

$$|I_1|=|\int_0^1\{1-\lambda-t^\beta\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho t^\rho]\}$$
$$\times LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right)$$
$$\times[ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt|$$
$$\leq\int_0^1\{|1-\lambda|+t^\beta\mathfrak{F}^{\sigma}_{\rho,\beta+1}[|w|(a_2^p-a_1^p)^\rho t^\rho]\}$$
$$\times|LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right)|$$
$$\times[ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt$$
$$=\int_0^1\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(a_2^p-a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|1-\lambda|\}$$
$$\times|LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right)|$$
$$\times[ta_1^p+(1-t)(\lambda_1 a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt$$

$$\leq [a_1^p + (1-\lambda_1)(a_2^p-\kappa)]^{\frac{1-p}{p}} \int_0^1 \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)} t^{\beta+\rho k} + |1-\lambda| \right\}$$
$$\times [|LME'(a_1)|\{t^s + (1-t)^s \lambda_1^s\} + (1-t)^s (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|]$$
$$\times \left[1 - t\frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p + (1-\lambda_1)(a_2^p-\kappa)}\right]^{\frac{1-p}{p}} dt$$
$$= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{[a_1^p + (1-\lambda_1)(a_2^p-\kappa)]^{\frac{p-1}{p}} \Gamma(\rho k+\beta+1)} \int_0^1 t^{\beta+\rho k} \times$$
$$[|LME'(a_1)|\{t^s + (1-t)^s \lambda_1^s\} + (1-t)^s (1-\lambda_1)^s LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})]$$
$$\times \left[1 - t\frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p + (1-\lambda_1)(a_2^p-\kappa)}\right]^{\frac{1-p}{p}} dt + \frac{|1-\lambda|}{[a_1^p + (1-\lambda_1)(a_2^p-\kappa)]^{\frac{p-1}{p}}}$$
$$\times \int_0^1 [|LME'(a_1)|\{t^s + (1-t)^s \lambda_1^s\} + (1-t)^s (1-\lambda_1)^s LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})]$$
$$\times \left[1 - t\frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p + (1-\lambda_1)(a_2^p-\kappa)}\right]^{\frac{1-p}{p}} dt$$
$$= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{[a_1^p + (1-\lambda_1)(a_2^p-\kappa)]^{\frac{p-1}{p}} \Gamma(\rho k+\beta+1)} \{|LME'(a_1)|\mathbb{B}(\beta+\rho k+s+1, 1)$$
$$\times {}_2F_1\left(\frac{p-1}{p}, \beta+\rho k+s+1; \beta+\rho k+s+2, \frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p + (1-\lambda_1)(a_2^p-\kappa)}\right)$$
$$+(|LME'(a_1)|\lambda_1^s + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|)\mathbb{B}(\beta+\rho k+1, s+1)$$
$$\times {}_2F_1\left(\frac{p-1}{p}, \beta+\rho k+1; \beta+\rho k+s+2, \frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p + (1-\lambda_1)(a_2^p-\kappa)}\right)\Bigg\}$$
$$+\frac{|1-\lambda|}{[a_1^p + (1-\lambda_1)(a_2^p-\kappa)]^{\frac{p-1}{p}}} \{|LME'(a_1)|\mathbb{B}(s+1, 1)$$
$$\times {}_2F_1\left(\frac{p-1}{p}, s+1; s+2, \frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p + (1-\lambda_1)(a_2^p-\kappa)}\right)$$
$$+(|LME'(a_1)|\lambda_1^s + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|)\mathbb{B}(1, s+1)$$
$$\times {}_2F_1\left(\frac{p-1}{p}, 1; s+2, \frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p + (1-\lambda_1)(a_2^p-\kappa)}\right)\Bigg\}. \quad (3.\ 32)$$

Again by $(s,p)-$convexity of $|LME'|$

$$\begin{aligned}
|I_2| &= |\int_0^1 (\mu - t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho t^\rho]) \\
&\quad \times LME'(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)))}dt \\
&\quad \times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt| \\
&\leq \int_0^1 \{|\mu| + t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[|w|(a_2^p - a_1^p)^\rho t^\rho]\} \\
&\quad \times |LME'\left(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)| \\
&\quad \times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \\
&= \int_0^1 \{\sum_{k=0}^\infty \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)} t^{\beta + \rho k} + |\mu|\} \\
&\quad \times |LME'\left(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)| \\
&\quad \times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \\
&\leq a_2^{1-p} \int_0^1 \left\{\sum_{k=0}^\infty \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)} t^{\beta + \rho k} + |\mu|\right\} \\
&\quad \times [|LME'(a_2)|(1-t)^s + t^s \lambda_1^s |LME'(a_2)| + t^s (1-\lambda_1)^s \\
&\quad \times |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|] \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{\frac{1-p}{p}} dt \\
&= \sum_{k=0}^\infty \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{a_2^{p-1}\Gamma(\rho k + \beta + 1)} \int_0^1 t^{\beta+\rho k}[|LME'(a_2)|(1-t)^s \\
&\quad + t^s\{\lambda_1^s |LME'(a_2)| + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|\}] \\
&\quad \times \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{\frac{1-p}{p}} dt + \frac{|\mu|}{a_2^{p-1}} \int_0^1 [|LME'(a_2)|(1-t)^s \\
&\quad + t^s\{\lambda_1^s |LME'(a_2)| + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|\}] \\
&\quad \times \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{\frac{1-p}{p}} dt \\
&= \sum_{k=0}^\infty \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{a_2^{p-1}\Gamma(\rho k + \beta + 1)} \{|LME'(a_2)| \mathbb{B}(\beta + \rho k + 1, 1 + s) \\
&\quad \times {}_2F_1\left(\frac{p-1}{p}, \beta + \rho k + 1; \beta + \rho k + s + 2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right) \\
&\quad + (|LME'(a_2)|\lambda_1^s + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|) \mathbb{B}(\beta + \rho k + s + 1, 1) \\
&\quad \times {}_2F_1\left(\frac{p-1}{p}, \beta + \rho k + s + 1; \beta + \rho k + s + 2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right)\} \\
&\quad + \frac{|\mu|}{a_2^{p-1}} \{|LME'(a_2)| \mathbb{B}(1, 1+s) {}_2F_1\left(\frac{p-1}{p}, 1; s+2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right) \\
&\quad + (|LME'(a_2)|\lambda_1^s + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|) \mathbb{B}(1+s, 1) \\
&\quad \times {}_2F_1\left(\frac{p-1}{p}, s+1; s+2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right)\}. \qquad (3.\ 33)
\end{aligned}$$

Combining the inequalities (3.31), (3.32), (3.33), then using the defining (2.17), (2.18), $(2.23)-(2.26)$ and (2.10) yields the desired inequality (3.30). □

Following result provides trapezoidal type inequality for Mittag-Leffler type convex functions.

Corollary 3.3. *Let $f : I_1 \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I_1°, interior of I_1, such that $|LME'|$ is $s-$convex for $s \in (0,1]$, then*

$$\left|\Psi_1^1\left(a_1,a_2,1,\frac{a_1+a_2}{2};\alpha\right)\right| \leq \Big[\Big\{\mathfrak{H}_0^{1,s}\left(a_1,a_2,0,1;\eta_4;0\right)-\mathfrak{L}^s\left(a_1,a_2,0,1;0;\eta_5\right)\Big\} \times(s+1)+\mathfrak{G}_s^0\left(a_1,a_2,0,1;1;0;\eta_5\right)\Big]\frac{(a_2-a_1)|LME'(a_1)|}{4(s+1)} \quad (3.\ 34)$$

provided that $\Psi_1^1\left(a_1,a_2,1,\frac{a_1+a_2}{2};\alpha\right)$ and $|LME'(x)|$ are, respectively, defined by (2.14) *and* (2.3).

Proof. The proof directly follows from Theorem 3.2 for $\lambda,\mu,\beta,p,\sigma(0)\to 1$; $w,\lambda_1\to 0$; $\kappa\to\frac{a_1^p+a_2^p}{2}$. □

Theorem 3.4. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, interior of I, $a_1,a_2 \in I^\circ$ with $a_1 < a_2$ such that $|LME'|^y$ is $(s,p)-$convex for $\mathbf{R} \ni p \neq 0$ and $\rho,\beta>0$; let $g(\xi)=\sqrt[p]{\xi}$, $\xi>0$, $(s,\lambda_1)\in(0,1]\times[0,1]$ and $y>1$ be such that $y=\frac{x}{x-1}$, then*

$$|\Psi_\lambda^\mu(a_1,a_2,p,\kappa;\alpha)| \leq \mathfrak{F}^{\sigma_2}_{\rho,\beta+1}[|w|(a_2^p-a_1^p)^\rho]+G_2(a_1,a_2,s,p,\lambda,\mu,\kappa). \quad (3.\ 35)$$

Proof. By Holder's inequality and $(s,p)-$convexity of $|LME'|^y$

$$\begin{aligned}|I_1| &= |\int_0^1\{1-\lambda-t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p-a_1^p)^\rho t^\rho]\}\\ &\quad\times LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right)\\ &\quad\times[ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt|\\ &\leq\int_0^1\{\sum_{k=0}^\infty\frac{\sigma(k)|w|^k(a_2^p-a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|1-\lambda|\}\\ &\quad\times|LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right)|\\ &\quad\times[ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{\frac{1-p}{p}}dt\end{aligned}$$

$$\leq \{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(a_2^p-a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)}$$

$$\times\sqrt[x]{\int_0^1 t^{x(\beta+\rho k)}[ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{x\frac{1-p}{p}}dt}$$

$$+|1-\lambda|\sqrt[x]{\int_0^1[ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{x\frac{1-p}{p}}dt}\}$$

$$\times\sqrt[y]{\int_0^1|LME'\left(\sqrt[p]{ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))}\right)|^y dt}$$

$$\leq \{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(a_2^p-a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)}$$

$$\times\sqrt[x]{\int_0^1 t^{x(\beta+\rho k)}[ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{x\frac{1-p}{p}}dt}$$

$$+|1-\lambda|\sqrt[x]{\int_0^1[ta_1^p+(1-t)(\lambda_1a_1^p+(1-\lambda_1)(a_1^p+a_2^p-\kappa))]^{x\frac{1-p}{p}}dt}\}$$

$$\times\{\int_0^1[|LME'(a_1)|^y\{t^s+(1-t)^s\lambda_1^s\}+(1-t)^s(1-\lambda_1)^s$$

$$\times|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^y]dt\}^{\frac{1}{y}}$$

$$=\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(a_2^p-a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x(p-1)}}$$

$$\times\sqrt[x]{\int_0^1 t^{x(\beta+\rho k)}\left[1-t\frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x\frac{1-p}{p}}dt}$$

$$+\frac{|1-\lambda|\sqrt[x]{\int_0^1\left[1-t\frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x\frac{1-p}{p}}dt}}{\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x(p-1)}}\}\{\int_0^1[|LME'(a_1)|^y$$

$$\{t^s+(1-t)^s\lambda_1^s\}+(1-t)^s(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^y]dt\}^{\frac{1}{y}}$$

$$=\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(a_2^p-a_1^p)^{k\rho}\sqrt[x]{\mathbb{B}(x\beta+xk\rho+1,1)}}{\Gamma(\rho k+\beta+1)\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x(p-1)}}\times\right.$$

$$\sqrt[x]{{}_2F_1\left(x\frac{p-1}{p},x\beta+xk\rho+1;x\beta+xk\rho+2,\frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right)}$$

$$\left.+\frac{|1-\lambda|\sqrt[x]{{}_2F_1\left(x\frac{p-1}{p},1;2,\frac{(1-\lambda_1)(a_2^p-\kappa)}{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right)}}{\left[\sqrt[p]{a_1^p+(1-\lambda_1)(a_2^p-\kappa)}\right]^{x(p-1)}}\right\}$$

$$\times\sqrt[y]{\frac{|LME'(a_1)|^y(1+\lambda_1^s)+(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p+a_2^p-\kappa})|^y}{s+1}}. \quad (3.\ 36)$$

Again by Holder's inequality and $(s,p)-$convexity of $|LME'|^y$

$$
\begin{aligned}
|I_2| = &| \int_0^1 (\mu - t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho t^\rho]) \\
&\times LME'(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}) \\
&\times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt| \\
\leq &\int_0^1 \{|\mu| + t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[|w|(a_2^p - a_1^p)^\rho t^\rho]\} \\
&\times |LME'(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))})| \\
&\times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \\
= &\int_0^1 \{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)} t^{\beta + \rho k} + |\mu|\} \\
&\times |LME'(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))})| \\
&\times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \\
\leq &\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)} \\
&\times \sqrt[x]{\int_0^1 t^{x(\beta+\rho k)} [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{x\frac{1-p}{p}} dt} \\
&+ |\mu| \sqrt[x]{\int_0^1 [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{x\frac{1-p}{p}} dt}\} \\
&\times \sqrt[y]{\int_0^1 |LME'(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))})|^y dt} \\
\leq &\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)} \\
&\times \sqrt[x]{\int_0^1 t^{x(\beta+\rho k)} [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{x\frac{1-p}{p}} dt} \\
&+ |\mu| \sqrt[x]{\int_0^1 [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{x\frac{1-p}{p}} dt}\} \\
&\times \{\int_0^1 [|LME'(a_2)|^y (1-t)^s + t^s \\
&\times |LME'(\sqrt[p]{\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa)})|^y dt]\}^{\frac{1}{y}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)} \\
&\times \sqrt[x]{\int_0^1 t^{x(\beta+\rho k)}[(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{x\frac{1-p}{p}} dt} \\
&+|\mu| \sqrt[x]{\int_0^1 [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{x\frac{1-p}{p}} dt}\} \\
&\times\{\int_0^1 [|LME'(a_2)|^y\{t^s\lambda_1^s + (1-t)^s\} \\
&+t^s(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^y]dt\}^{\frac{1}{y}} \\
&= \{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)a_2^{x(p-1)}} \times \\
&\sqrt[x]{\int_0^1 t^{x(\beta+\rho k)} \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{x\frac{1-p}{p}} dt} \\
&+ \frac{|\mu|}{a_2^{x(p-1)}} \sqrt[x]{\int_0^1 \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{x\frac{1-p}{p}} dt}\}\{\int_0^1 [|LME'(a_2)|^y \\
&\times\{t^s\lambda_1^s + (1-t)^s\} + t^s(1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^y]dt\}^{\frac{1}{y}} \\
&= \left\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(a_2^p - a_1^p)^{k\rho} \sqrt[x]{\mathbb{B}(x\beta + xk\rho + 1, 1)}}{\Gamma(\rho k + \beta + 1)a_2^{x(p-1)}} \times \right. \\
&\sqrt[x]{{}_2F_1\left(x\frac{p-1}{p}, x\beta + xk\rho + 1; x\beta + xk\rho + 2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right)} \\
&\left. + \frac{|\mu| \sqrt[x]{{}_2F_1\left(x\frac{p-1}{p}, 1; 2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right)}}{a_2^{x(p-1)}} \right\} \\
&\times \sqrt[y]{\frac{|LME'(a_2)|^y(1+\lambda_1^s) + (1-\lambda_1)^s|LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^y}{s+1}}. \qquad (3.\ 37)
\end{aligned}
$$

Combining the inequalities (3.31), (3.36)-(3.37), then using the defining (2.19), (2.20), (2.23), (2.24), (2.26) and (2.10) yields the desired inequality (3.35). □

Following result provides some mid point type inequality for Mittag-Leffler type convex functions.

Corollary 3.5. *Let $f : I_1 \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I_1°, interior of I_1, such that $|LME'|^y$ is $s-$convex for $y > 1$ such that $y = \frac{x}{x-1}$ and $s \in (0,1]$, then*

$$\left|\Psi_0^0\left(a_1, a_2, 1, \frac{a_1+a_2}{2}; \alpha\right)\right| \leq \frac{(a_2-a_1)|LME'(a_1)|}{4\sqrt[y]{s+1}}\left[\sqrt[x]{\mathfrak{H}_0^{x,0}(a_1,a_2,0,1;\eta_4;0)} + \sqrt[x]{\mathfrak{H}_0^{x,0}(a_1,a_2,0,1;0;\eta_5)} + \sqrt[x]{\mathfrak{G}_0^0(a_1,a_2,1;x;\eta_4;0)}\right] \quad (3.\ 38)$$

provided that $\Psi_0^0\left(a_1, a_2, 1, \frac{a_1+a_2}{2}; \alpha\right)$ and $|LME'(x)|$ are, respectively, defined by (2.15) and (2.3).

Proof. The proof directly follows from Theorem 3.4 for $\beta, p, \sigma(0) \to 1$; $\lambda, \mu, w, \lambda_1 \to 0$; $\kappa \to \frac{a_1^p + a_2^p}{2}$. □

Theorem 3.6. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, interior of I, $a_1, a_2 \in I^\circ$ with $a_1 < a_2$ such that $|LME'|^x$ is $(s,p)-$convex for $\mathbf{R} \ni p \neq 0$ $\rho, \beta > 0$, $x \geq 1$; let $g(\xi) = \sqrt[p]{\xi}$, $\xi > 0$ and $(s, \lambda_1) \in (0,1] \times [0,1]$, then*

$$|\Psi_\lambda^\mu(a_1, a_2, p, \kappa; \alpha)| \leq \mathfrak{F}_{\rho,\beta+1}^{\sigma_3}[|w|(a_2^p - a_1^p)^\rho] + G_3(a_1, a_2, s, p, \lambda, \mu, \kappa). \quad (3.\ 39)$$

Proof. By $(s,p)-$convexity of $|LME'|^x$ and power-mean inequality which is

$$\int_a^b |\mathbf{f}(\mathfrak{x})\mathbf{g}(\mathfrak{x})|dx \leq \left(\int_a^b |\mathbf{f}(\mathfrak{x})|d\mathfrak{x}\right)^{\frac{x-1}{x}} \left(\int_a^b |\mathbf{f}(\mathfrak{x})||\mathbf{g}(\mathfrak{x})|^x d\mathfrak{x}\right)^{\frac{1}{x}},$$

$$\begin{aligned}
|I_1| = &\left| \int_0^1 \{1 - \lambda - t^\beta \mathfrak{F}_{\rho,\beta+1}^{\sigma}[w(a_2^p - a_1^p)^\rho t^\rho]\} \right. \\
&\times LME'\left(\sqrt[p]{ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right) \\
&\left. \times [ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \right| \\
\leq &\int_0^1 \{|1-\lambda| + t^\beta \mathfrak{F}_{\rho,\beta+1}^{\sigma}[|w|(a_2^p - a_1^p)^\rho t^\rho]\} \\
&\times \left|LME'\left(\sqrt[p]{ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)\right| \\
&\times [ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \\
= &\int_0^1 \{\sum_{k=0}^\infty \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)} t^{\beta + \rho k} + |1-\lambda|\} \\
&\times \left|LME'\left(\sqrt[p]{ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)\right| \\
&\times [ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)}$$
$$\times \{\int_0^1 t^{\beta+\rho k}[ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}$$
$$\times \{\int_0^1 t^{\beta+\rho k}[ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} \times$$
$$|LME'\left(\sqrt[p]{ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)|^x dt\}^{\frac{1}{x}}$$
$$+|1-\lambda|\{\int_0^1 [ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}$$
$$\times \{\int_0^1 [ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} \times$$
$$|LME'\left(\sqrt[p]{ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)|^x dt\}^{\frac{1}{x}}$$
$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)}$$
$$\times \{\int_0^1 t^{\beta+\rho k}[ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}$$
$$\times \{\int_0^1 t^{\beta+\rho k}[ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}}$$
$$\times [t^s |LME'(a_1)|^x + (1-t)^s \{\lambda_1^s |LME'(a_1)|^x$$
$$+(1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}] dt\}^{\frac{1}{x}}$$
$$+|1-\lambda|\{\int_0^1 [ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}$$
$$\times \{\int_0^1 [ta_1^p + (1-t)(\lambda_1 a_1^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} \times$$
$$[t^s |LME'(a_1)|^x + (1-t)^s \{\lambda_1^s |LME'(a_1)|^x + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}] dt\}^{\frac{1}{x}}$$
$$= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)\left[\sqrt[p]{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{p-1}}$$
$$\times \{\int_0^1 t^{\beta+\rho k}\left[1 - t\frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}$$
$$\times \{\int_0^1 [t^{\beta+\rho k+s} |LME'(a_1)|^x + t^{\beta+\rho k}(1-t)^s \{\lambda_1^s |LME'(a_1)|^x$$
$$+(1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}]\left[1 - t\frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{\frac{1-p}{p}} dt\}^{\frac{1}{x}}$$
$$+\frac{|1-\lambda|\{\int_0^1 \left[1 - t\frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}}{\left[\sqrt[p]{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{p-1}}\{\int_0^1 \left[1 - t\frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{\frac{1-p}{p}}$$
$$\times [t^s |LME'(a_1)|^x + (1-t)^s \{\lambda_1^s |LME'(a_1)|^x$$
$$+(1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}] dt\}^{\frac{1}{x}}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1)\left[\sqrt[p]{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{p-1}} \{\mathbb{B}(\beta + k\rho + 1, 1) \\
&\times {}_2F_1\left(\frac{p-1}{p}, \beta + k\rho + 1; \beta + k\rho + 2, \frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right)\}^{\frac{x-1}{x}} \\
&\times \{{}_2F_1\left(\frac{p-1}{p}, \beta + k\rho + s + 1; \beta + k\rho + s + 2, \frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right) \\
&\times \mathbb{B}(\beta + k\rho + s + 1, 1)|LME'(a_1)|^x + \{\lambda_1^s |LME'(a_1)|^x \\
&+ (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\} \\
&\times {}_2F_1\left(\frac{p-1}{p}, \beta + k\rho + s + 1; \beta + k\rho + s + 2, \frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right) \\
&\times \mathbb{B}(\beta + k\rho + 1, 1 + s)\}^{\frac{1}{x}} \\
&+ \frac{|1-\lambda|\{{}_2F_1\left(\frac{p-1}{p}, 1; 2, \frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right)\}^{\frac{x-1}{x}}}{\left[\sqrt[p]{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{p-1}} \\
&\times \{{}_2F_1\left(\frac{p-1}{p}, s + 1; s + 2, \frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right) \\
&\times \mathbb{B}(s + 1, 1)|LME'(a_1)|^x \\
&+ \{\lambda_1^s |LME'(a_1)|^x + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\} \times \\
&{}_2F_1\left(\frac{p-1}{p}, 1; s + 2, \frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right) \mathbb{B}(1, s + 1)\}^{\frac{1}{x}}. \qquad (3.\ 40)
\end{aligned}
$$

Again by $(s,p)-$convexity of $|LME'|^x$ and power-mean inequality:

$$
\begin{aligned}
|I_2| &= |\int_0^1 \{\mu - t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(a_2^p - a_1^p)^\rho t^\rho]\} \\
&\quad \times LME'\left(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right) \\
&\quad \times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt| \\
&\le \int_0^1 \{|\mu| + t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[|w|(a_2^p - a_1^p)^\rho t^\rho]\} \\
&\quad \times |LME'\left(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)| \\
&\quad \times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 \{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)} t^{\beta+\rho k} + |\mu|\} \\
&\quad \times |LME'\left(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)| \\
&\quad \times [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt \\
&\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)} \\
&\quad \times \{\int_0^1 t^{\beta+\rho k} [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}} \\
&\quad \times \{\int_0^1 t^{\beta+\rho k} [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} \\
&\quad \times |LME'\left(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)|^x dt\}^{\frac{1}{x}} \\
&\quad + |\mu| \{\int_0^1 [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}} \\
&\quad \times \{\int_0^1 [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} \\
&\quad \times |LME'\left(\sqrt[p]{(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))}\right)|^x dt\}^{\frac{1}{x}} \\
&\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k+\beta+1)} \\
&\quad \times \{\int_0^1 t^{\beta+\rho k} [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}} \\
&\quad \times \{\int_0^1 t^{\beta+\rho k} [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} \times \\
&\quad [(1-t)^s |LME'(a_2)|^x + t^s \{\lambda_1^s |LME'(a_2)|^x + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}] dt\}^{\frac{1}{x}} \\
&\quad + |\mu| \{\int_0^1 [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}} \\
&\quad \times \{\int_0^1 [(1-t)a_2^p + t(\lambda_1 a_2^p + (1-\lambda_1)(a_1^p + a_2^p - \kappa))]^{\frac{1-p}{p}} \\
&\quad \times [(1-t)^s |LME'(a_2)|^x + t^s \{\lambda_1^s |LME'(a_2)|^x \\
&\quad + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}] dt\}^{\frac{1}{x}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho} \{\int_0^1 t^{\beta+\rho k} \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}}{\Gamma(\rho k + \beta + 1) a_2^{p-1}} \\
&\quad \times \{ \int_0^1 [t^{\beta+\rho k}(1-t)^s |LME'(a_2)|^x + t^{\beta+\rho k + s}\{\lambda_1^s |LME'(a_2)|^x \\
&\quad + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}] \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{\frac{1-p}{p}} dt\}^{\frac{1}{x}} \\
&\quad + \frac{|\mu| \{\int_0^1 \left[1 - t\frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right]^{\frac{1-p}{p}} dt\}^{\frac{x-1}{x}}}{a_2^{p-1}} \{ \int_0^1 \left[1 - t\frac{(1-\lambda_1)(a_2^p - \kappa)}{a_1^p + (1-\lambda_1)(a_2^p - \kappa)}\right]^{\frac{1-p}{p}} \\
&\quad \times [(1-t)^s |LME'(a_2)|^x + t^s \{\lambda_1^s |LME'(a_2)|^x \\
&\quad + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\}] dt\}^{\frac{1}{x}} \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (a_2^p - a_1^p)^{k\rho}}{\Gamma(\rho k + \beta + 1) a_2^{p-1}} \{\mathbb{B}(\beta + k\rho + 1, 1) \\
&\quad \times {}_2F_1\left(\frac{p-1}{p}, \beta + k\rho + 1; \beta + k\rho + 2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right)\}^{\frac{x-1}{x}} \\
&\quad \times \{ {}_2F_1\left(\frac{p-1}{p}, \beta + k\rho + 1; \beta + k\rho + s + 2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right) \\
&\quad \times \mathbb{B}(\beta + k\rho + 1, 1 + s) |LME'(a_2)|^x \\
&\quad + \{\lambda_1^s |LME'(a_2)|^x + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\} \times \\
&\quad {}_2F_1\left(\frac{p-1}{p}, \beta + k\rho + s + 1; \beta + k\rho + s + 2, \frac{(1-\lambda_1)(k - a_1^p)}{a_2^p}\right) \\
&\quad \times \mathbb{B}(\beta + k\rho + s + 1, 1)\}^{\frac{1}{x}} + \frac{|\mu| \{ {}_2F_1\left(\frac{p-1}{p}, 1; 2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right)\}^{\frac{x-1}{x}}}{a_2^{p-1}} \\
&\quad \times \{ {}_2F_1\left(\frac{p-1}{p}, 1; s+2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right) \mathbb{B}(1, s+1) |LME'(a_2)|^x \\
&\quad + \{\lambda_1^s |LME'(a_2)|^x + (1-\lambda_1)^s |LME'(\sqrt[p]{a_1^p + a_2^p - \kappa})|^x\} \times \\
&\quad {}_2F_1\left(\frac{p-1}{p}, 1 + s; s+2, \frac{(1-\lambda_1)(\kappa - a_1^p)}{a_2^p}\right) \mathbb{B}(1 + s, 1)\}^{\frac{1}{x}}. \qquad (3.\ 41)
\end{aligned}
$$

Combining the inequalities (3.31) and (3.40)-(3.41), then using the defining (2.21) − (2.24), (2.26) and (2.10) yields the desired inequality (3.39). □

Following result provides Simpson type inequality for Mittag-Leffler type convex functions.

Corollary 3.7. *Let $f : I_1 \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I_1°, interior of I_1, such that $|LME'|^x$ is $s-$convex for $x \geq 1$ and $s \in (0,1]$, then*

$$\begin{aligned}\left|\Psi_{\frac{1}{3}}^{\frac{1}{3}}\left(a_1, a_2, 1, \frac{a_1+a_2}{2}; \alpha\right)\right| \leq \frac{(a_2-a_1)|LME'(a_1)|}{12\sqrt[x]{s+1}} \\ \times \Bigg\{3\sqrt[x]{s+1}\Bigg[\sqrt[x]{\mathfrak{H}_0^{1,s}(a_1,a_2,0,1;\eta_4;0)[\mathfrak{H}_0^{1,0}(a_1,a_2,0,1;\eta_4;0)]^{x-1}} \\ + \sqrt[x]{[\mathfrak{H}_0^{1,0}(a_1,a_2,0,1;0;\eta_5)]^{x-1}\mathfrak{L}^s(a_1,a_2,0,1;0;\eta_5)}\Bigg]+ \\ 2\sqrt[x]{\mathfrak{G}_s^s(a_1,a_2,1;1;\eta_4;0)[\mathfrak{G}_0^0(a_1,a_2,1;1;\eta_4;0)]^{x-1}} \\ + \sqrt[x]{[\mathfrak{G}_0^0(a_1,a_2,1;1;0;\eta_5)]^{x-1}\mathfrak{G}_s^0(a_1,a_2,1;1;0;\eta_5)}\Bigg]\Bigg\}, \quad (3.\ 42)\end{aligned}$$

provided that $\Psi_{\frac{1}{3}}^{\frac{1}{3}}\left(a_1, a_2, 1, \frac{a_1+a_2}{2}; \alpha\right)$ and $|LME'(x)|$ are, respectively, defined by (2.16) *and* (2.3).

Proof. The proof directly follows from Theorem 3.6 for $\beta, p, \sigma(0) \to 1$; $w, \lambda_1 \to 0$; $\lambda, \mu \to \frac{1}{3}$; $\kappa \to \frac{a_1^p+a_2^p}{2}$. □

Remark 3.8. *For other choices of the parameters in above Theorems, some more results can be derived which are being omitted.*

4. APPLICATIONS

Let $a_1, a_2 \in \mathbf{R}^+$, set of positive reals, then the some following means are

$$A(a_1,a_2) := \frac{a_1+a_2}{2}; \quad G(a_1,a_2) := \sqrt{a_1a_2}; \quad LP(a_1,a_2) := \frac{a_2\ln a_2 - a_1 \ln a_1}{a_2-a_1}.$$

Consider, $f(x) = \ln\left(\frac{1}{x}\right) = -\ln x$. Then, obviously, $f(x)$ is exp-convex and in this case (2.2) reduces to $LME\left(\frac{a_1+a_2}{2}\right) = f\left(\frac{a_1+a_2}{2}\right)$ and

$$\begin{aligned}\int_{a_1}^{a_2} LME(t)dt &= \int_{a_1}^{a_2}\left[\ln\left(\frac{a_1+a_2}{2}\right) - \ln t - \ln(a_1+a_2-t)\right]dt \\ &= (a_2-a_1)\ln\left(\frac{a_1+a_2}{2}\right) - a_2\ln a_2 + a_1 \ln a_1 - a_1 + a_2 \\ &\quad -a_2\ln a_1 + a_1 \ln a_2 + \int_{a_1}^{a_2}\left[1 - \frac{a_1+a_2}{a_1+a_2-t}\right]dt \\ &= (a_2-a_1)\ln\left(\frac{a_1+a_2}{2}\right) - 2[a_2\ln a_2 - a_1\ln a_1] + 2(a_2-a_1) \\ &= (a_2-a_1)[\ln A(a_1,a_2) - 2LP(a_1,a_2)+2]\end{aligned}$$

$$\ln\frac{E_\alpha(f(a_1)) + E_\alpha(f(a_2))}{2} = \ln\left(\frac{a_1+a_2}{2a_1a_2}\right) = \ln A(a_1,a_2) - 2\ln G(a_1,a_2).$$

$$\mathfrak{L}^s(a_1,a_2,0,1;0;\eta_5) = \frac{1}{(s+2)(s+1)}; \quad \mathfrak{H}_0^{1,s}(a_1,a_2,0,1;\eta_4;0) = \frac{1}{s+2}$$

$$\mathfrak{H}_0^{x,0}(a_1,a_2,0,1;\eta_4;0)=\mathfrak{H}_0^{x,0}(a_1,a_2,0,1;0;\eta_5)=\frac{1}{x+1};\ \mathfrak{H}_0^{1,0}(a_1,a_2,0,1;0;\eta_5)=\frac{1}{2}$$

$$\mathfrak{G}_0^0(a_1,a_2,1;x;\eta_4;0)=\mathfrak{G}_0^0(a_1,a_2,1;1;0;\eta_5)=\mathfrak{G}_s^s(a_1,a_2,1;1;\eta_4;0)$$
$$=\mathfrak{G}_s^0(a_1,a_2,1;1;0;\eta_5)=1.$$

In the light of the above discussion, some consequences of the inequalities (3.34), (3.38) and (3.42) are the followings:

$$|LP(a_1,a_2)-\ln G(a_1,a_2)-1|\le\frac{(a_2-a_1)^2}{4a_1a_2(s+2)}$$

$$|LP(a_1,a_2)-\ln A(a_1,a_2)-1|\le\frac{(a_2-a_1)^2[2+\sqrt[x]{x+1}]}{8a_1a_2\sqrt[x]{x+1}\sqrt[y]{s+1}}$$

$$|3LP(a_1,a_2)-3-\ln G(a_1,a_2)-2\ln A(a_1,a_2)|\le\frac{3\sqrt[x]{2}(a_2-a_1)^2}{16a_1a_2}\left[\frac{1+\sqrt[x]{s+1}}{\sqrt[x]{(s+1)(s+2)}}+\frac{6}{\sqrt[x]{2}}\right].$$

5. Conclusion

This research work was carried out using the new defined concept of Mittag Leffler type convexity, generalizing already defined notion so called as exponential type convexity. New multi-parameters fractional integral identity has been proved. As a result, new unified fractional portmanteau versions of Hermite-Hadamard type, trapezoidal type, mid point type and Simpson type inequalities are emerged. Furthermore, some applications to special means have been discussed of the derived results.

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